MATHEMATICS MAGAZINE

CONTENTS

VOLUME 39 • JANUARY 1966 • NUMBER 1

MATHEMATICS MAGAZINE

ROY DUBISCH, *Editor*

EDITORIAL CORRESPONDENCE should be sent to the Editor, ROY DUBISCH, Department of Mathematics, University of Washington, Seattle, Washington 98105. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript, and authors should keep a complete copy. Figures should be drawn on separate sheets in India ink and of a suitable size for photographing.

NOTICE OF CHANGE OF ADDRESS and other subscription correspondence should be sent to the Executive Director, H. M. GEHMAN, Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

ADVERTISING CORRESPONDENCE should be addressed to Raoul Hailpern Mathematical Association of America, SUNY at Buffalo (University of Buffalo), Buffalo, New York 14214.

The MATHEMATICS MAGAZINE is published by the Mathematical Association of America at Buffalo, New York, bi-monthly except July-August. Ordinary subscriptions are: 1 year \$3.00; 2 years \$5.75; 3 years \$8.50; 4 years \$11.00; 5 years \$13.00. Members of the Mathematical Association of America and of Mu Alpha Theta may subscribe at the special rate of 2 years for \$5.00. Single copies are 65¢.

Second class postage paid at Buffalo, New York and additional mailing offices.

MATHEMATICS MAGAZINE

Published by

THE MATHEMATICAL ASSOCIATION OF AMERICA

(INCORPORATED)

A JOURNAL
OF COLLEGIATE MATHEMATICS

VOLUME 39

1966

Published at

MENASHA, WIS., AND BUFFALO, N. Y.

BOOLEAN MATRICES AND SWITCHING NETS

WAI-KAI CHEN, Ohio University

1. Introduction. The problem of synthesis of switching systems is one of ever increasing importance to modern technology; it arises in the design of all sorts of pushbutton devices. The recent appearance of a number of papers [1–7] on switching circuit theory in which boolean matrices have been used makes it appropriate to discuss certain of these applications and their significances. In the present paper, a short review of boolean matrix algebra which has arisen in switching circuit theory is given, though no attempt is made to give more than an outline of certain phases of the subject. Later, a few possible extensions are discussed.

2. Boolean matrices. A boolean matrix [3] is simply a matrix over a boolean algebra, i.e., a rectangular array of elements from a boolean algebra. These arrays are subject to appropriate rules of operation, some of which are analogous to the rules of operation for ordinary matrices, whereas others reflect the boolean character of the elements.

Let $\tilde{B}$ be a boolean algebra of at least two elements, and M be the set of all boolean matrices of order $m\times n$. The following notation will be used to indicate intersection, union, complementation, and inclusion: Suppose $\tilde{B}=\{a, b, c, \cdots\}$. We will write $a\cap b$ for intersection, $a\cup b$ for union, $\bar{a}$ for complementation, $a\leqq b$ for inclusion.

In the set $M=\{A, B, C, \cdots\}$ where $a_{ij}, b_{ij}, c_{ij}, \cdots, (i=1, 2, \cdots, m; j=1, 2, \cdots, n)$ are elements of corresponding matrices $A, B, C, \cdots$, respectively, we define

$$\begin{array}{llll} A = B & \text{iff } a_{ij} = b_{ij} & \text{for all } i \text{ and } j \\ A = B\cap C & \text{iff } a_{ij} = b_{ij}\cap c_{ij} & \text{for all } i \text{ and } j \\ A = B\cup C & \text{iff } a_{ij} = b_{ij}\cup c_{ij} & \text{for all } i \text{ and } j \\ A = \overline{B} & \text{iff } a_{ij} = \bar{b}_{ij} & \text{for all } i \text{ and } j \\ A \leqq B & \text{iff } a_{ij} \leqq b_{ij} & \text{for all } i \text{ and } j. \end{array}$$

If 0 and 1 are the zero and universal elements of $\tilde{B}$, then in M the zero, identity, and universal matrices O, W, and I are defined as:

$$O = [0],\ W = [e_{ij}] \text{ where } e_{ij} = 0 \quad \text{for } i \neq j$$
$$= 1 \quad \text{for } i = j$$

and $I=[1]$ for all i and j. The transpose of A, denoted by A^t, is defined by $A^t=[d_{ij}]$ where $d_{ij}=a_{ji}$ for all i and j.

It is easy to show that under these definitions M forms a boolean algebra. A multiplication may be defined as follows:

$$AB = \left[\bigcup_{k=1}^{q} (a_{ik}\cap b_{kj})\right]_{p\times r}$$

where A is a boolean matrix of order $p \times q$ and B is of order $q \times r$. If this operation is used instead of $A \cap B \in M$ and $m=n$, then M does not form a boolean algebra with respect to the operations "$\cdot$", "$\cup$" and "$-$". However, under these definitions and the given multiplication, M forms a semi-group with identity element.

In analogy to ordinary matrix theory, Luce [3] has defined concepts of symmetry, skew symmetry, orthogonal and inverse of a boolean matrix. A boolean matrix A is said to be *symmetric* iff $A^t \cap \bar{A} = O$; A is *skew-symmetric* iff $A^t \cap A = O$; the *inverse* of A, if it exists, denoted by A^{-1}, is such that $AA^{-1} = A^{-1}A = W$; and A is *orthogonal* iff it has an inverse which is A^t. With these definitions some interesting properties are found. Those discovered by Luce [3] will be restated as follows without proof.

THEOREM 1. *A boolean matrix has an inverse iff it is orthogonal.*

THEOREM 2. *Any boolean matrix can be uniquely decomposed into the disjoint union of a symmetric and a skew-symmetric matrix.*

Two observations could be given here for these definitions and theorems formulated by Luce which will be stated as Corollary 1 and Corollary 2.

COROLLARY 1. *A boolean matrix A is symmetric iff $A = A^t$ or $A^t \cup \bar{A} = \mathrm{I}$.*

COROLLARY 2. *A is involutory ($AA = W$) iff A is both orthogonal and symmetric.*

In addition to those in the literature [3, 6, 8] yet other definitions, parallel to one in ordinary matrix theory, may be considered. Since these new terms do lead to some interesting results, their existence is therefore justified.

DEFINITION 1. *A boolean matrix A is said to be tranjugate iff $A^t \cup A = \mathrm{I}$.*

LEMMA 1. *Given any A, $B \in M$, then $A \cup B = \mathrm{I}$ iff $\bar{A} \leqq B$.*

COROLLARY 3. *A boolean matrix A is tranjugate iff $\bar{A}^t \leqq A$.*

COROLLARY 4. *If a boolean matrix A is tranjugate, then the diagonal elements must be* 1.

These results are especially useful since a switching matrix has been defined [6] as any boolean matrix having all 1's down the main diagonal whereas in a boolean matrix A the elements down the main diagonal are arbitrary over $\tilde{B}$. For a skew-symmetric matrix the elements along the main diagonal are 0 since $a_{ii} \cap a_{ii} = 0$ for all i.

THEOREM 3. *Any boolean matrix can be uniquely decomposed into the joint intersection of a symmetric and a tranjugate matrix.*

Proof. For any decomposition $A = S \cap T$ subject to $S^t \cap \bar{S} = O$, $T^t \cup T = I$, and $S \cup T = I$, one has

$$\begin{aligned} A \cup A^t &= (S \cap T) \cup (S \cap T)^t = ((S \cap T) \cup S^t) \cap ((S \cap T) \cup T^t) \\ &= (S \cup S^t) \cap (S^t \cup T) \cap (S \cup T^t) \cap (T \cup T^t) = I \cap S = S \end{aligned}$$

$$
\begin{aligned}
A \cup \overline{A}^t &= (S \cap T) \cup (\overline{S}^t \cup \overline{T}^t) = (S \cup \overline{S}^t \cup \overline{T}^t) \cap (T \cup \overline{S}^t \cup \overline{T}^t) \\
&= I \cap (T \cup \overline{S} \cup \overline{T}^t) \\
&= T \cup \overline{S} && \text{(by Corollary 3)} \\
&= T && (\text{since } S \cup T = I \text{ iff } \overline{S} \leqq T, \text{ implies } \overline{S} \cup T = T)
\end{aligned}
$$

so that at most one such decomposition can exist.

Moreover, the choices of S, T suggested by the uniqueness argument above do in fact always give a decomposition of the required type. For, given any boolean matrix A and defining $S = A \cup A^t$ and $T = A \cup \overline{A}^t$ one obtains

$$
\begin{aligned}
A &= A \cup (A^t \cap \overline{A}^t) = (A \cup A^t) \cap (A \cup \overline{A}^t) = S \cap T \\
S^t \cap \overline{S} &= (A \cup A^t) \cap (A \cap \overline{A}^t) \\
&= (A \cap \overline{A} \cap \overline{A}^t) \cup (A^t \cap \overline{A} \cap \overline{A}^t) = 0 \\
T^t \cup T &= (A^t \cup \overline{A}) \cup (A \cup \overline{A}^t) = A \cup \overline{A} \cup A^t \cup \overline{A}^t = I \\
S \cup T &= I.
\end{aligned}
$$

This completes the proof of the theorem.

3. Switching matrices and matrices associated with combinational circuits. As mentioned in the previous section, any boolean matrix which has all 1's down the main diagonal will be called a switching matrix. A switching circuit with p output terminals is said to be *combinational* if the outputs are a unique function of the inputs.

From an abstract point of view, any switching circuit can be represented by a weighted, directed, linear graph (or simply digraph). A *directed linear graph* G consists of a set V of elements called nodes together with a set E of ordered pairs of the form (i, j), i and $j \in V$, called the edges of the graph; the node i is called the *initial node* and node j the *terminal node*. A directed graph S is a *subgraph* of G if the nodes and edges of S are nodes and edges respectively of G and if each edge of S has the same initial node and the same terminal node in S as in G. If A is a subset of V, the *sectional graph* $G[A]$ of G defined by A is the subgraph whose node set is A and whose edges are all those edges in G which connect two nodes in A. When $A = V$ the sectional graph is G itself. A *directed path* p_{ij} is a subgraph of the form $p_{ij} = (i, k_1)(k_1, k_2)(k_2, k_3) \cdots (k_m, j)$, where i, j and k_t (for $t = 1, \cdots, m$) are nodes in V. It is required that all the nodes of p_{ij} shall be distinct.

In a combinational circuit, after a brief operate-time, the state f_{ij} of the "connection" between the nodes i and j depends only on the combination of values assumed by the input variables, and hence may be represented as a boolean function. In order to describe the terminal behavior of the digraph associated with such a circuit, two matrices are defined [6] as follows:

$$\text{Output matrix } F = [f_{ij}]_{p \times p}$$

where f_{ij} is the boolean function between nodes i and j, and p is the number of the output terminals. Since a node is always connected to itself, f_{ii} is defined to be 1 for each i.

(Primitive) Connection matrix $C = [c_{ij}]_{n\times n}$ $(c_{ii} = 1$ for each $i)$,

where c_{ij} is the boolean function associated with the edge (i, j), $i \neq j$, of the corresponding digraph and n is the number of nodes in the digraph. Nodes associated with output terminals are called *output nodes*; otherwise, *nonoutput nodes.*

If the boolean algebra $\tilde{B}$ contains only two elements, denoted by 0 and 1, then c_{ij} represents the "connection" between nodes i and j. This symbol has the value 0 if there is no connection at all and 1 if there is a short circuit, but otherwise it is the symbol denoting a single contact or a union of such symbols. Properties and techniques discussed here are more general and can be applied to any boolean algebra.

Certainly any switching matrix may be interpreted as a connection matrix of a combinational switching circuit or a digraph. However, the question of how to characterize an output matrix is more interesting, and the answer is given by Hohn and Schissler [6].

THEOREM 4. *The necessary and sufficient condition that a switching matrix* F *be an output matrix is that* $F^2 = F$.

Another useful tool in the analysis of switching circuits is the or-determinant [2] of a square boolean matrix A of order m, the row expansion of which is defined as

$$\det A = \bigcup_{(j)} a_{1j_1}a_{2j_2} \cdots a_{mj_m}$$

where the symbol (j) means that the union is to be extended over all permutations $j_1j_2 \cdots j_m$ of the integers $1, 2, \cdots, m$.

Similarly, the *cofactor* of a_{ij}, denoted by $\det A_{ij}$, of $\det A$ is defined as the or-determinant of the matrix obtained from A by striking out the ith row and jth column of A. The *adjoint matrix of* A, denoted by $\{A\}$, is the matrix $[d_{ij}]$ of order m such that $d_{ij} = \det A_{ji}$; for the present case $a_{ii} = 1$ for all i.

These definitions are like the definitions of the determinant of a matrix over the complex field except that "$\cup$" replaces "$+$" and no sign variations appear. From this it follows that the purely combinational properties of determinants of matrices over the complex field will also apply in the boolean case. Those properties which depend on the signs of the terms may not be carried over in the same way, however; e.g.,

1. $\det A^t = \det A$.
2. The interchange of two lines of a matrix leaves the determinant invariant.
3. If A_i, $i = 1, 2, \cdots, m$ are the columns of A, then

$$\det [A_1, A_2, \cdots, \beta A_k, \cdots, A_m] = \beta \det A$$

$$\det [A_1 \cup A_i, A_2, \cdots, A_m] = (\det A) \cup (\det [A_i, A_2, \cdots, A_m])$$

where β is a scalar.

4. Laplace expansion also holds in this case except there are no sign variations.

For the use in the analysis of switching circuits the following result is obtained.

THEOREM 5. *If C is the connection matrix of a switching circuit, and F is the output matrix, then $F = \{C\}$.*

4. The analysis and synthesis of combinational circuits. The basic problem of analysis of combinational circuits is the determination of the relation between any given connection matrix and the corresponding output matrix. To accomplish this, Hohn and Schissler [**6**] showed first how to obtain from a given circuit an equivalent circuit using one less non-output node in the formation of the connection matrix. This operation (the star-mesh transformation [**6**]) is repeated until there are no non-output nodes in the accounting.

Matrixwise, this operation proceeds as follows: To remove a non-output node r, one joins to entry c_{ij} of C the intersection of the entry c_{ir} by the entry c_{rj}, thereafter deleting row r and column r from C. It is interesting to note this node removal process is very similar to Chio's method for the reduction of the order of a determinant in ordinary matrix theory. This method is very useful because it can be reversed, i.e., node insertion. In other words if a new line is to be placed in a reduced matrix, every row and every column must contain a same element, and the intersection of these two elements must in turn be contained in one entry.

A natural question to ask at this point is, "Is it possible to generalize the single-node removal process to the process of multiple-node removal?" Shekel [**8**] in an unpublished note first obtained this generalization in an algebraic form. However, Shekel's process can be accomplished topologically as follows.

Let C be the connection matrix (not necessarily symmetric) of a given digraph G; let C be partitioned in such a way that

$$C = [c_{ij}]_{n\times n} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}_{n\times n}$$

where C_{11} is a square submatrix of order p which corresponds to the output nodes of G.

For convenience, a mapping function f is defined from the edges (i, j) of G into a boolean algebra $\tilde{B}$ such that $f((i, j)) = c_{ij}$ for all edges in G where $c_{ij} \in \tilde{B}$. This definition is extended to any nonempty subgraph R of G as follows:

$$f(R) = \bigcap f((u_1, u_2))$$

where the intersection runs over all edges (u_1, u_2) in R.

Next let

$$C_{t_1t_2} = [c_{ij}^{t_1t_2}] \quad t_1, t_2 = 1, 2, \qquad \{C_{22}\} = [m_{ij}]_{(n-p)\times(n-p)}$$

and

$$F = [f_{ij}]_{p\times p}$$

where $\{C_{22}\}$ is the adjoint matrix of C_{22}.

Yoeli [8] has shown that if F is the corresponding output matrix of order p, then $F=\{C_{11}\cup C_{12}\{C_{22}\}C_{21}\}$. Suppose $C_{11}\cup C_{12}\{C_{22}\}C_{21}=[d_{ij}]$, then

$$d_{ij} = \overset{11}{c_{ij}} \cup \left[\bigcup_{k,t=1}^{n-p} (\overset{12}{c_{ik}} \cap m_{kt} \cap \overset{21}{c_{tj}}) \right].$$

Since m_{kt} is the switching function from node k to node t of the circuit corresponding to C_{22}, it follows that

$$m_{kt} = \bigcup_{p'_{kt}} f(p'_{kt})$$

where p'_{kt} is a directed path from node k to node t in the sectional graph $G[A]$ where A is the node set corresponding to the submatrix C_{22}, and the union is taken over all possible $p'_{kt}\in G[A]$. Therefore

$$\begin{aligned} d_{ij} &= \bigcup_{p_{ij}} f(p_{ij}) \quad \text{for } i \neq j \\ &= 1 \qquad\qquad\quad \text{for } i = j \end{aligned}$$

where p_{ij} is a directed path from node i to node j in the sectional graph $G[A\cup i\cup j]$, where $A\cup i\cup j$ is the set union of the nodes i, j, and the node set A; and the union is taken over all possible $p_{ij}\in G[A\cup i\cup j]$. One may now state

THEOREM 6. *Let V be the node set of a given digraph G, and V_m be any subset of the node set which corresponds to the non-output nodes of V. Then the digraph G_r and G have the same output matrix where G_r is derived from G by the following procedure.*

(a) *Remove $G[V_m]$ from G, i.e., remove all nodes in V_m and also all edges incident to and coming from any node in V_m. Edges which do not previously exist in G may be considered as edges which map to zero, i.e., $f((i, j))=0$ for all (i, j) not in G where f is the mapping function of G.*

(b)
$$\begin{aligned} f_r((i,j)) &= \bigcup_{p_{ij}} f(p_{ij}) \quad \text{for } i \neq j \\ &= 1 \qquad\qquad\quad \text{for } i = j \end{aligned}$$

for all $i, j\in(V-V_m)$ where f_r is the mapping function of G_r; $(V-V_m)$ represents the elements contained in V but not in V_m; p_{ij} is a direct path from nodes i to j in the sectional graph $G[V_m\cup i\cup j]$ where $V_m\cup i\cup j$ is the set union of the nodes i, j and the node set V_m; and the union is taken over all possible $p_{ij}\in G[V_m\cup i\cup j]$.

It is interesting to note that if V_m contains only one node, the above theorem reduces to the star-mesh transformation in Hohn and Schissler's paper [6].

At this point, it is obvious that the topological reduction process not only displays in a very intuitive manner the causal relationships among the variables of the system under study, but also shows that the process is independent of the labelling of the nodes.

Example 1. Consider the circuit shown in Fig. 1(a). The dotted part is the sectional graph to be removed. The reduced circuit is shown in Fig. 1(b). (For convenience, "$\cup$," "$\cap$," and "$-$" are replaced by "$+$" juxtaposition, and "$'$" in all the figures, respectively.)

Evidently this process can be easily applied to sequential circuits to give the corresponding multiple "state removal" algorithm with minor modifications.

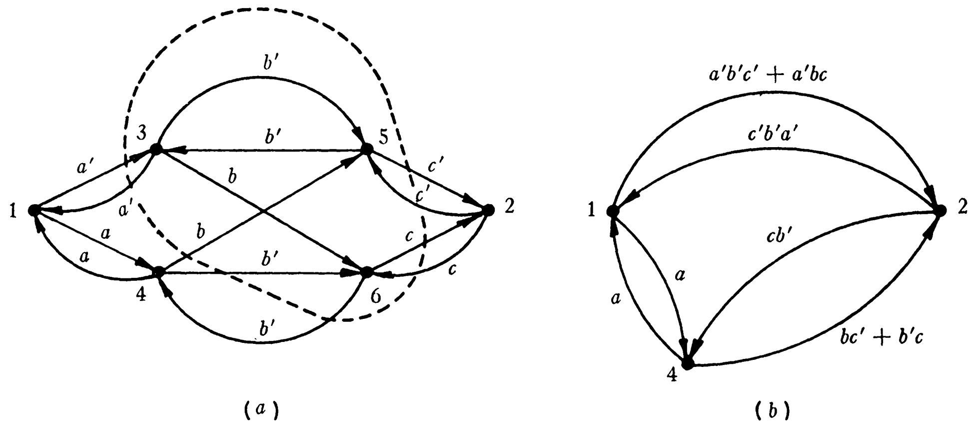


FIG. 1 (a) Example for multiple-node removal.
(b) The corresponding reduced circuit of (a).

If $G[A]$ is the sectional graph to be removed from a state diagram G where A is a set of states and if there exist self-loops in $G[A]$, then the process shown in Fig. 2 must be used in order to eliminate all such self-loops (1 is used as identity for multiplication but $1\cup b\neq 1$). k is a nonnegative integer.

After all the self-loops having been removed, Theorem 6 now can be applied to obtain the corresponding reduced state diagram.

Example 2. Consider the state diagram of Fig. 3(a). The dotted part is the sectional graph to be removed. Fig. 3(b) is the corresponding reduced state diagram.

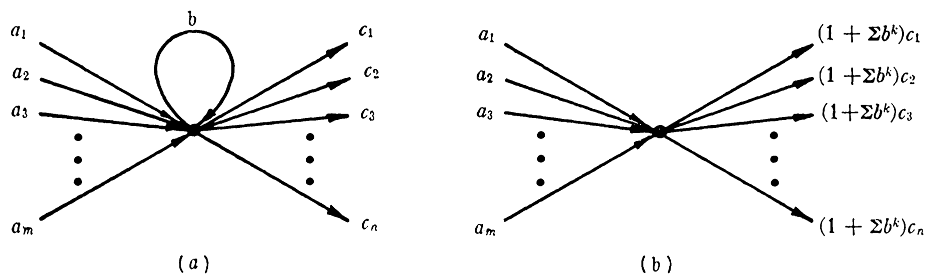


FIG. 2 (a) A node with a self-loop.
(b) With the self-loop removed.

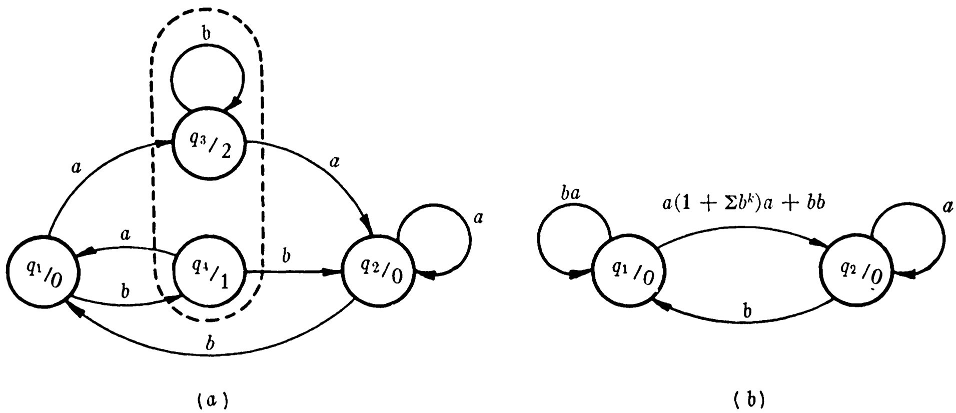


FIG. 3. Example for multiple state removal.

5. Conclusions. In this paper the basic properties of boolean matrices have been discussed and their applications to the analysis of both combinational relay and sequential circuits were studied. As mentioned earlier, no attempt is made to give more than an outline of certain phases of the subject. For a more extensive treatment, reference should be made to the various papers and works in the field.

References

1. A. G. Lunts, The application of Boolean matrix algebra to the analysis and synthesis of relay contact networks, Doklady Akademii Nauk, SSSR, 70 (1950) 421–423.

2. W. Semon, Matrix theory of switching networks, Thesis, Harvard University, The Computation Lab., 1954.

3. R. D. Luce, Note on Boolean matrix theory, Proc. Amer. Math. Soc., 3 (1952) 382–388.

4. B. I. Aranovich, The use of matrix methods in problems of structural analysis of relay-contact networks, Avtomatika i Telemekhanika, SSSR, 10 (1949) 437–451.

5. M. L. Tsetlin, Application of matrix calculus to the synthesis of relay-contact networks, Doklady Akademii Nauk, SSSR, No. 3, 86 (1952) 525–528.

6. F. E. Hohn and L. R. Schissler, Boolean matrices in the design of combinational switching circuits, Bell System Tech. J., 34 (Jan., 1955) 177–202.

7. F. E. Hohn, A matrix method for the design of relay circuits, IRE Trans. on Circuit Theory, Vol. CT-2, (June, 1955) 154–161.

8. M. Yoeli, The theory of switching nets, IRE Trans. on Circuit Theory, Vol. CT-6 (special supplement) (May, 1959) 152–157.

SOME RADICAL AXES ASSOCIATED WITH THE CIRCUMCIRCLE

D. MOODY BAILEY, Princeton, West Virginia

Part III

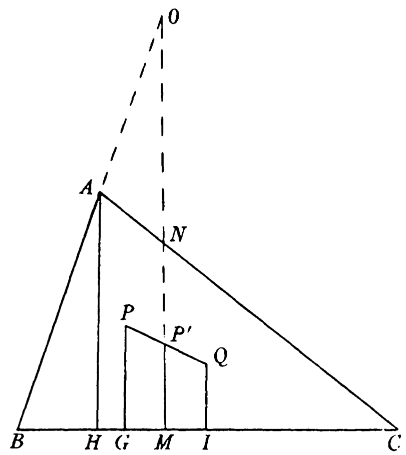


FIG. 5.

Allow P and Q to be the symmedian point and circumcenter respectively of triangle ABC. The circle constructed on PQ as diameter is known as the Brocard circle of the triangle. P', the center of the Brocard circle, will evidently lie at the midpoint of segment PQ. From vertex A and points P, P', Q drop perpendiculars to side BC thereby determining points H, G, M, I (Fig. 5). Extend MP' to meet sides CA and AB at respective points N and O. If DEF is the cevian triangle of symmedian point P, it is known that $BD/DC=c^2/b^2$, $CE/EA=a^2/c^2$, $AF/FB=b^2/a^2$. Also [7]

$$\frac{BG}{GC}=\frac{2a^2(AE/EC)+a^2+c^2-b^2}{2a^2(AF/FB)+a^2+b^2-c^2},$$

and replacing AE/EC and AF/FB by the quantities c^2/a^2 and b^2/a^2, this becomes

$$\frac{BG}{GC}=\frac{a^2+3c^2-b^2}{a^2+3b^2-c^2}.$$

Using this value of BG/GC in the identity $BG=a(BG/GC)/(BG/GC+1)$, we find that

$$BG=\frac{a(a^2+3c^2-b^2)}{2(a^2+b^2+c^2)}.$$

Since P' bisects PQ, it is evident that $BM=(BG+BI)/2$. Using the value just obtained for BG, and replacing BI by $a/2$, we determine that

$$BM = \frac{a(a^2 + 2c^2)}{2(a^2 + b^2 + c^2)} \cdot$$

Then

$$MC = a - BM = \frac{a(a^2 + 2b^2)}{2(a^2 + b^2 + c^2)}$$

and

$$\frac{BM}{MC} = \frac{a^2 + 2c^2}{a^2 + 2b^2} \cdot \tag{1}$$

Now $HC = b \cos C = (a^2 + b^2 - c^2)/2a$ by the law of cosines. Also

$$HM = HC - MC = \frac{b^4 - c^4}{2a(a^2 + b^2 + c^2)}, \quad \text{and} \quad \frac{HM}{MC} = \frac{b^4 - c^4}{a^2(a^2 + 2b^2)} \cdot$$

Ratio HM/MC is equivalent to ratio AN/NC and we may write

$$\frac{CN}{NA} = \frac{a^2(a^2 + 2b^2)}{b^4 - c^4} \cdot \tag{2}$$

Ratio values (1) and (2) have been determined for line MNO and the theorem of Menelaus now shows that

$$\frac{AO}{OB} = \frac{c^4 - b^4}{a^2(a^2 + 2c^2)} \cdot \tag{3}$$

Consequently, the ratios associated with line MNO have been calculated.

Suppose that segment PQ be extended to meet sides BC, CA, AB at points M', N', O' respectively. Cevian ratios $AF/FB = b^2/a^2$ and $AE/EC = c^2/a^2$, associated with the symmedian point, are substituted in the equation $(AF/FB) \cdot (BO'/O'A) + (AE/EC) \cdot (CN'/N'A) = 1$, [2] to give $(b^2/a^2) \cdot (BO'/O'A) + (c^2/a^2) \cdot (CN'/N'A) = 1$. Similarly the ratios of the circumcenter give [8]

$$\frac{b^2}{a^2}\left(\frac{a^2 + c^2 - b^2}{b^2 + c^2 - a^2}\right)\frac{BO'}{O'A} + \frac{c^2}{a^2}\left(\frac{a^2 + b^2 - c^2}{b^2 + c^2 - a^2}\right)\frac{CN'}{N'A} = 1.$$

The simultaneous solution of these two equations shows that

$$\frac{BO'}{O'A} = \frac{a^2}{b^2}\left(\frac{a^2 - c^2}{b^2 - c^2}\right) \quad \text{and} \quad \frac{CN'}{N'A} = \frac{a^2}{c^2}\left(\frac{a^2 - b^2}{c^2 - b^2}\right) \cdot$$

$BM'/M'C$ is then found through the use of the theorem of Menelaus. So line PQ, through the symmedian point and circumcenter, has the ratio values

$$\frac{BM'}{M'C} = \frac{c^2}{b^2}\left(\frac{c^2 - a^2}{b^2 - a^2}\right) \tag{4}$$

$$\frac{CN'}{N'A} = \frac{a^2}{c^2}\left(\frac{a^2 - b^2}{c^2 - b^2}\right), \tag{5}$$

$$(6) \qquad \frac{AO'}{O'B} = \frac{b^2}{a^2}\left(\frac{b^2 - c^2}{a^2 - c^2}\right).$$

Now P' is the point of intersection of lines MNO and $M'N'O'$. If $D'E'F'$ be the cevian triangle of point P', we may write

$$\frac{BD'}{D'C} = -\frac{BO/OA - BO'/O'A}{CN/NA - CN'/N'A},$$

with similar values existing for $CE'/E'A$ and $AF'/F'B$ [6]. By substituting in the right members of these equations from (1), (2), (3), (4), (5), (6) the following result is obtained:

THEOREM 9. *P' is the center of the Brocard circle of triangle ABC, with $D'E'F'$ its cevian triangle. The ratio values for point P' are*

$$\frac{BD'}{D'C} = \frac{c^2}{b^2}\left(\frac{2a^2b^2 + a^2c^2 + b^2c^2 - c^4}{2a^2c^2 + a^2b^2 + b^2c^2 - b^4}\right), \qquad \frac{CE'}{E'A} = \frac{a^2}{c^2}\left(\frac{2b^2c^2 + a^2b^2 + a^2c^2 - a^4}{2a^2b^2 + b^2c^2 + a^2c^2 - c^4}\right),$$

$$\frac{AF'}{F'B} = \frac{b^2}{a^2}\left(\frac{2a^2c^2 + b^2c^2 + a^2b^2 - b^4}{2b^2c^2 + a^2c^2 + a^2b^2 - a^4}\right).$$

This conclusion has been derived primarily for use in the proof of the next theorem. However, Theorem 9 is important in its own right and the reader should observe that the method used in securing this result may be utilized to determine ratio values for the midpoint of any segment PQ where the ratios associated with points P and Q are known. In fact, it is possible to derive a general theorem for the ratio values of P' in terms of ratios BD/DC, CE/EA, AF/FB, the sides of triangle ABC, and the ratios connected with point Q.

Suppose that P'' is now the isogonal conjugate of point P' of Theorem 9. If $D''E''F''$ be the cevian triangle of P'', we must have

$$\frac{BD''}{D''C} = \frac{2a^2c^2 + a^2b^2 + b^2c^2 - b^4}{2a^2b^2 + a^2c^2 + b^2c^2 - c^4}.$$

The equation $(BD'/D'C)(BD''/D''C) = c^2/b^2$, true of every pair of isogonal rays through vertex A, will then be satisfied. In a similar manner the ratios $CE''/E''A$ and $AF''/F''B$ are determined. Remembering that the ratios c^2/b^2, a^2/c^2, b^2/a^2 are associated with the symmedian point of the triangle, we have, in the light of Theorems 2 and 8B, this relationship:

THEOREM 8C. *Let P'' be the isogonal conjugate of the center of the Brocard circle of triangle ABC, with $D''E''F''$ its cevian triangle. Let P, the symmedian point of the triangle, have DEF for its cevian triangle. The three radical axes of the circumcircle and circles ADD'', BEE'', CFF'' meet sides BC, CA, AB at respective points M, N, O. Line MNO is the radical axis of the pedal circle of the Brocard points and circumcircle ABC.*

Suppose that P is a point in the plane of triangle ABC having ratio values $BD/DC=c^4/b^4$, $CE/EA=a^4/c^4$, $AF/FB=b^4/a^4$. It is not difficult to show that point P is then the isogonal conjugate of the isotomic conjugate of the symmedian point of the triangle. Furthermore, point P lies on the line connecting the symmedian point and circumcenter and is the point of intersection of the tangents to the Brocard circle at the Brocard points [**9**]. Through the use of Theorems 2, 9, 8B another interesting relationship is derived.

THEOREM 8D. *P is a point in the plane of triangle ABC having cevian triangle DEF such that $BD/DC=c^4/b^4$, $CE/EA=a^4/c^4$, $AF/FB=b^4/a^4$. P' is the isotomic conjugate of the center of the Brocard circle, with $D'E'F'$ its cevian triangle. The three radical axes of the circumcircle and circles ADD', BEE', CFF' meet sides BC, CA, AB at respective points M, N, O. Line MNO is the radical axis of the pedal circle of the Brocard points and circumcircle ABC.*

The converse of this theorem, as well as that of Theorem 8C, may be used to fix the center of the Brocard circle.

A special theorem needed in obtaining our final result with respect to radical axes will now be derived. Let points D, E, F be selected in any manner on sides BC, CA, AB of triangle ABC. (Rays AD, BE, CF need not be concurrent and DEF need not be the pedal triangle of any point with respect to triangle ABC.) Construct circle DEF to meet sides BC, CA, AB again at respective points D', E', F'. Can values for $BD'/D'C$, $CE'/E'A$, $AF'/F'B$ be determined in terms of ratios BD/DC, CE/EA, AF/FB and sides a, b, c of triangle ABC? If such expressions exist they must evidently reduce to the forms known to hold when rays AD, BE, CF are concurrent [**5**, Theorem 2].

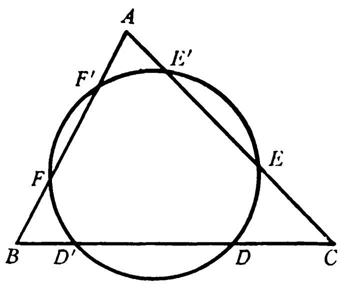


FIG. 6.

Since points E, E', F', F are cyclic (Fig. 6), we may write $AF\cdot AF'=AE\cdot AE'$, or $(AF/AE)(AF'/AE')=1$. Now

$$\frac{AF}{AE}=\frac{FA}{EA}=\frac{c}{b}\cdot\frac{b}{EA}\cdot\frac{FA}{c}=\frac{c}{b}\cdot\frac{\dfrac{b}{EA}}{\dfrac{c}{FA}}=\frac{c}{b}\left[\frac{\dfrac{CE+EA}{EA}}{\dfrac{BF+FA}{FA}}\right]=\frac{c}{b}\left[\frac{\dfrac{CE}{EA}+1}{\dfrac{BF}{FA}+1}\right].$$

In similar fashion

$$\frac{AF'}{AE'} = \frac{c}{b}\left[\frac{\dfrac{CE'}{E'A}+1}{\dfrac{BF'}{F'A}+1}\right].$$

The equation $(AF/AE)(AF'/AE')=1$ then becomes

$$\frac{c^2}{b^2}\left[\frac{\dfrac{CE}{EA}+1}{\dfrac{BF}{FA}+1}\right]\left[\frac{\dfrac{CE'}{E'A}+1}{\dfrac{BF'}{F'A}+1}\right] = 1. \tag{7}$$

Applying the same procedure with respect to vertex B and the cyclic points F', F, D', D, we obtain

$$\frac{a^2}{c^2}\left[\frac{\dfrac{AF}{FB}+1}{\dfrac{CD}{DB}+1}\right]\left[\frac{\dfrac{AF'}{F'B}+1}{\dfrac{CD'}{D'B}+1}\right] = 1. \tag{8}$$

Finally, vertex C and cyclic points D', D, E, E' give

$$\frac{b^2}{a^2}\left[\frac{\dfrac{BD}{DC}+1}{\dfrac{AE}{EC}+1}\right]\left[\frac{\dfrac{BD'}{D'C}+1}{\dfrac{AE'}{E'C}+1}\right] = 1. \tag{9}$$

Solving equation (7) for $CE'/E'A$, we obtain

$$\frac{CE'}{E'A} = \frac{\dfrac{b^2}{c^2}\left(\dfrac{BF}{FA}+1\right)\left(\dfrac{BF'}{F'A}+1\right)}{\left(\dfrac{CE}{EA}+1\right)} - 1.$$

In similar fashion the solution of equation (9) gives

$$\frac{CE'}{E'A} = \frac{\left(\dfrac{AE}{EC}+1\right)}{\dfrac{b^2}{a^2}\left(\dfrac{BD}{DC}+1\right)\left(\dfrac{BD'}{D'C}+1\right) - \left(\dfrac{AE}{EC}+1\right)}.$$

The right members of these two equations are placed equal to each other and the resulting equation is solved for $BF'/F'A$. This yields

$$(10)\quad \frac{BF'}{F'A}=\frac{\left(\frac{BD}{DC}+1\right)\left(\frac{BD'}{D'C}+1\right)\left[c^2\left(\frac{CE}{EA}+1\right)-b^2\left(\frac{BF}{FA}+1\right)\right]+a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)}{b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)\left(\frac{BD'}{D'C}+1\right)-a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)}.$$

Equation (8) is now solved for $BF'/F'A$ and we get

$$\frac{BF'}{F'A}=\frac{a^2\left(\frac{AF}{FB}+1\right)}{c^2\left(\frac{CD}{DB}+1\right)\left(\frac{CD'}{D'B}+1\right)-a^2\left(\frac{AF}{FB}+1\right)}.$$

In this equation $(CD'/D'B+1)$ may be replaced by $(BD'/D'C+1)/(BD'/D'C)$ to obtain

$$(11)\quad \frac{BF'}{F'A}=\frac{a^2\left(\frac{AF}{FB}+1\right)\frac{BD'}{D'C}}{c^2\left(\frac{CD}{DB}+1\right)\left(\frac{BD'}{D'C}+1\right)-a^2\left(\frac{AF}{FB}+1\right)\frac{BD'}{D'C}}.$$

Let the right members of (10) and (11) be equated. Cross multiplication then yields a rather complicated quadratic equation in the ratio $BD'/D'C$. The quantity $c^2(BD/DC+1)$ is found to be a factor of this result. However, for this to be true, $(CD/DB+1)$ must be replaced by $(BD/DC+1)CD/DB$ in a couple of instances. Also, some other changes such as $(AE/EC+1)$ and $(BF/FA+1)$ being replaced by $(CE/EA+1)AE/EC$ and $(AF/FB+1)BF/FA$ are made. Eventually our result is in the form $A(BD'/D'C)^2+B(BD'/D'C)+C=0$, where

$$A=a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)+b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{CD}{DB}+1\right)\left(\frac{CE}{EA}+1\right),$$

$$B=2b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-2c^2\left(\frac{CD}{DB}+1\right)\left(\frac{CE}{EA}+1\right)-a^2(X),$$

$$C=b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)\frac{CD}{DB}-c^2\left(\frac{CD}{DB}+1\right)\left(\frac{CE}{EA}+1\right).$$

The last term in the above expression for B, represented by $a^2(X)$, has the value

$$(12)\quad a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)\left(\frac{BF}{FA}\cdot\frac{AE}{EC}\cdot\frac{CD}{DB}-1\right).$$

Should rays AD, BE, CF be concurrent, the theorem of Ceva gives $(BF/FA)\cdot(AE/EC)\cdot(CD/DB)=1$, and the last factor of $a^2(X)$ equals zero. The term $a^2(X)$ then vanishes. Let us assume that points D, E, F have been selected so that rays AD, BE, CF are not concurrent. Then (12) must be carried as the final term in the expression represented by B.

The two roots of the quadratic equation $A(BD'/D'C)^2+B(BD'/D'C)+C=0$ are $BD'/D'C=[-B\pm\sqrt{(B^2-4AC)}]/2A$. Substituting the given values of A, B, C in this quadratic formula yields

$$\frac{BD'}{D'C}=\frac{a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)\frac{CD}{DB}-b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)+c^2\left(\frac{CD}{DB}+1\right)\left(\frac{CE}{EA}+1\right)}{a^2\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)+b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{CD}{DB}+1\right)\left(\frac{CE}{EA}+1\right)},$$

or $BD'/D'C=-1$.

The actual computation of the two roots through use of the quadratic formula turns out to be an extremely complicated process. To avoid possible frustration we shall show how the correctness of the two roots may be verified.

The equation $A(BD'/D'C)^2+B(BD'/D'C)+C=0$ is written in the longer form with A, B, C replaced by the values previously given. Then $BD'/D'C$ is replaced by -1 and the left member rather easily reduces to zero. This proves that -1 is a root of the equation and, as a consequence, $(BD'/D'C+1)$ must be a factor of the left member of the equation. If the first value of $BD'/D'C$ is a root of the equation, then we must have $(BD'/D'C-Y)(BD'/D'C+1)=0$. Replace Y by the first root obtained, carry out the indicated multiplication, and the previously derived equation $A(BD'/D'C)^2+B(BD'/D'C)+C=0$ will be obtained. In this way the reader may verify that the two given roots are correct even though substitution in the quadratic formula has not been employed.

Let the numerator and the denominator of the first root be multiplied by BD/DC so that $BD'/D'C$ is obtained in a better form. $CE'/E'A$ and $AF'/F'B$ are computed in a similar manner to give

THEOREM 10. *Let D, E, F be three points chosen in any manner on sides BC, CA, AB of triangle ABC. Circle DEF is constructed to meet these sides again at respective points D', E', F'. Then*

$$\frac{BD'}{D'C}=\frac{a^2\left(\frac{BF}{FA}+1\right)\left(\frac{AE}{EC}+1\right)-b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)+c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)}{a^2\frac{BD}{DC}\left(\frac{AF}{FB}+1\right)\left(\frac{CE}{EA}+1\right)+b^2\left(\frac{BD}{DC}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{BD}{DC}+1\right)\left(\frac{CE}{EA}+1\right)},$$

$$\frac{CE'}{E'A}=\frac{b^2\left(\frac{CD}{DB}+1\right)\left(\frac{BF}{FA}+1\right)-c^2\left(\frac{CE}{EA}+1\right)\left(\frac{CD}{DB}+1\right)+a^2\left(\frac{CE}{EA}+1\right)\left(\frac{AF}{FB}+1\right)}{b^2\frac{CE}{EA}\left(\frac{BD}{DC}+1\right)\left(\frac{AF}{FB}+1\right)+c^2\left(\frac{CE}{EA}+1\right)\left(\frac{CD}{DB}+1\right)-a^2\left(\frac{CE}{EA}+1\right)\left(\frac{AF}{FB}+1\right)},$$

$$\frac{AF'}{F'B}=\frac{c^2\left(\frac{AE}{EC}+1\right)\left(\frac{CD}{DB}+1\right)-a^2\left(\frac{AF}{FB}+1\right)\left(\frac{AE}{EC}+1\right)+b^2\left(\frac{AF}{FB}+1\right)\left(\frac{BD}{DC}+1\right)}{c^2\frac{AF}{FB}\left(\frac{CE}{EA}+1\right)\left(\frac{BD}{DC}+1\right)+a^2\left(\frac{AF}{FB}+1\right)\left(\frac{AE}{EC}+1\right)-b^2\left(\frac{AF}{FB}+1\right)\left(\frac{BD}{DC}+1\right)}.$$

The roots $BD'/D'C = CE'/E'A = AF'/F'B = -1$ are found to be without meaning except when points D, E, F happen to be collinear.

When DEF is the cevian triangle of point P, it is seen that Ceva's theorem allows us to replace ratio BD/DC by $(BF/FA)(AE/EC)$. The first term in the denominator of $BD'/D'C$ may then be reduced to $a^2(BF/FA+1)(AE/EC+1)$. Ratio $BD'/D'C$ of Theorem 10 then becomes equivalent to a known result [**5**, Theorem 2]. Similar comments apply to the ratios $CE'/E'A$ and $AF'/F'B$.

As an example of the use of Theorem 10, suppose that the rays AD, BE, CF are drawn through the centroid, positive Brocard point, and negative Brocard point of triangle ABC so that $BD/DC = 1$, $CE/EA = a^2/b^2$, $AF/FB = c^2/a^2$.

THEOREM 10A. *Rays AD, BE, CF pass through the centroid, positive Brocard point, and negative Brocard point respectively of triangle ABC. Circle DEF is constructed to meet sides BC, CA, AB again at points D', E', F' and*

$$\frac{BD'}{D'C} = \frac{b^2(a^2 + c^2)(a^2 - b^2) + 2c^4(a^2 + b^2)}{c^2(a^2 + b^2)(a^2 - c^2) + 2b^4(a^2 + c^2)},$$

$$\frac{CE'}{E'A} = \frac{c^2(a^2 + b^2)(a^2 - c^2) + 2b^4(a^2 + c^2)}{c^2(a^2 + c^2)(b^2 - a^2) + 2c^4(a^2 + b^2)},$$

$$\frac{AF'}{F'B} = \frac{b^2(a^2 + b^2)(c^2 - a^2) + 2b^4(a^2 + c^2)}{b^2(a^2 + c^2)(a^2 - b^2) + 2c^4(a^2 + b^2)}.$$

For Theorem 10A rays AD, BE, CF are nonconcurrent, as are rays AD', BE', CF', but the ratio values for points D', E', F' have been easily determined. A variety of other examples will doubtlessly occur to the reader interested in applications of Theorem 10.

By Theorem 1 the radical axis of circle DEF and circumcircle ABC meets BC at M so that $BM/MC = -(BD/DC)(BD'/D'C)$. Using the value of $BD'/D'C$ given in Theorem 10, we find that ratio BD/DC is again a factor of the denominator of $BD'/D'C$. After BM/MC has been determined we are somewhat surprised to find that it is identical to that given in Theorem 6. Evidently the concurrence, or nonconcurrence, of rays AD, BE, CF has no effect on ratio values associated with the radical axis of circles DEF and ABC.

THEOREM 11. *Let the points D, E, F be chosen in any manner on sides BC, CA, AB of triangle ABC. The radical axis of circles DEF and ABC meets sides BC, CA, AB at respective points M, N, O. Ratios BM/MC, CN/NA, AO/OB have the values recorded in Theorem 6.*

The preceding theorem is seen to solve completely the problem of determining the radical axis of the circumcircle and any other circle that meets the sides of triangle ABC. Theorems 6 and 8 thus become special cases of the general result expressed in Theorem 11. When dealing with the pedal circle, one may use Theorem 8 or may compute the ratios BG/GC, CH/HA, AI/IB [**7**] and then allow these results to replace the ratios BD/DC, CE/EA, AF/FB of Theorem 11 (Theorem 6). It is suggested that the reader check Theorem 11 using the ratios $BD/DC = (a^2+c^2-b^2)/(a^2+b^2-c^2)$, $CE/EA = 1$, $AF/FB = 1$ in the expressions for BM/MC, CN/NA, AO/OB. Point D is the foot of the altitude from vertex A to side BC, while points E and F are the midpoints of sides CA and AB. Circle DEF thus becomes the nine point circle of triangle ABC and rays AD, BE, CF are nonconcurrent. Theorem 11 (Theorem 6) will reveal that the radical axis of the circumcircle and nine point circle is the orthic axis as stated in Theorem 6A.

This study must now be brought to a conclusion. Other facts which the author had hoped to include must await future discussion. An urge to record theorems dealing with the conic sections has been present throughout the preparation of this material. For example, we may return to Theorem 2 and allow point P to be fixed but let P' traverse a fixed line in the plane of triangle ABC. Line MNO of Theorem 2 will then envelop a conic which touches the sides of triangle ABC at known points. In similar fashion theorems dealing with the conic sections parallel Theorems 3, 4, etc.

We have observed that Theorem 11 completely solves the problem of determining the radical axis for circles ABC and DEF where D, E, F are points selected in any manner on the sides of triangle ABC. Suppose that we now eliminate the circumcircle and select in any fashion two triads of points D, E, F and G, H, I on the sides of the triangle. Can ratio values then be determined for the radical axis of circles DEF and GHI in terms of ratios BD/DC, CE/EA, AF/FB, the sides of triangle ABC, and ratios BG/GC, CH/HA, AI/IB? The answer is in the affirmative and this discovery has led to a multitude of new facts. For instance, the well-known theorem of Feuerbach states thatt he nine-point circle of a triangle is tangent to the incircle and each of the three excircles. We have been able to write the ratio values for each of the four points of contact of these circles. (We sometimes wonder whether Feuerbach was acquainted with these facts.) Furthermore, ratio values for the radical axis of the nine-point circle and each of these circles has been recorded. It should be of interest to the reader to learn that the radical axis (common tangent) of the nine point circle and incircle meets the sides of triangle ABC so that $BM/MC=(b-a)/(c-a)$, $CN/NA=(c-b)/(a-b)$, $AO/OB=(a-c)/(b-c)$. Perhaps we shall have an opportunity to consider these matters more fully at some future time.

Finally, it is the hope of the author that the reader will feel inclined to further investigate some of the configurations associated with the theorems that have been given. Applications of the general results are numerous, interesting, and sometimes rather amazing.

Thanks are extended to the referee for suggestions concerning the preparation and simplification of this manuscript.

Credit for Theorem 2A, found in Part I of this article, should be given to Paul D. Thomas (see solution to problem No. 432, proposed by the latter, in National Math. Mag., 16 (1942) 306–307).

References

2. D. M. Bailey, A triangle theorem, School Science and Mathematics, 60 (1960) 281–4.
5. ———, Circle associate of a given point, this MAGAZINE, 37 (1964) 224–6.
6. ———, Point of intersection of triangle transversals, this MAGAZINE, 37 (1964) 331–3.
7. ———, On pedal ratios, this MAGAZINE, 38 (1965) 128–130.
8. ———, Problem Department, School Science and Mathematics, 61 (1961) 62.
9. R. A. Johnson, Modern geometry, 1st ed., Houghton Mifflin, New York, p. 282.

SOME RESULTS PERTAINING TO FERMAT'S CONJECTURE

G. J. SIMMONS, Sandia Corporation, Albuquerque, New Mexico

The Diophantine equation

$$x^n + y^n = z^n, \tag{1}$$

where x, y, z, and n are positive integers, is one of the most celebrated problems in number theory. It has become known as Fermat's Conjecture or Last Theorem since in the margin of his copy of Bachet's edition of Diophantos' Arithmetica, Fermat asserted without proof that he had discovered a marvelous demonstration that no solutions exist for $n>2$ and integers x, y and z all different from zero. The investigation of this problem has been a rich one, for example leading to the theory of ideals in the hands of Kummer. This paper collects several results obtainable by purely elementary means from an equivalent expression for the Fermat equation discovered by G. Reis in 1959.

Obviously from Equation (1) and the restriction of x, y and z to positive nonzero integers, $z>x$ and $z>y$. But the binomial expansion of $(x+y)^n$ as $x^n+y^n+[nx^{n-1}y+ \cdots +nxy^{n-1}]$ permits the following inequality to be given since the bracketed expression is greater than zero for $n>1$:

$$(x + y)^n > x^n + y^n = z^n.$$

Therefore, $x+y>z$. Introduce an arbitrary integer K on the right side of this expression to make an equality, i.e.,

$$x + y = z + K. \tag{2}$$

Then $y-K=z-x>0$, since $z>x$. Therefore, $y>K$, and by the same argument one finds $x>K$. It follows that since $z>x$ and $z>y$, that $z>K$ also. Convert these three inequalities to equalities by the introduction of three new, and for the moment, undetermined variables a, b and c.

$$\begin{aligned} x &= K + a \\ y &= K + b \\ z &= K + c. \end{aligned} \tag{3}$$

Introducing the forms given by (3) into Equation (2) shows that $c=a+b$, which completes the derivation of the Reis equivalent expression for the Fermat Equation (1):

$$(K + a)^n + (K + b)^n = (K + a + b)^n. \tag{4}$$

We shall first prove that $a \neq b$ for any solution. Assume $a=b$:

$$(K + a)^n + (K + a)^n = 2(K + a)^n = (K + 2a)^n. \tag{5}$$

Since the right hand term of (5) must be even, K is necessarily even. Let $K=2K'$; then (5) becomes:

$$(2K' + a)^n = 2^{n-1}(K' + a)^n. \tag{6}$$

But this in turn requires the left hand term of (6) to be even; hence $a=2a'$, and we obtain

$$2(K'+a')^n=(K'+2a')^n \tag{7}$$

which is the same form as (5). Therefore by appeal to the method of infinite descent, it is seen that there can be no solution for which $a=b$, and it is proven that

$$a\neq b. \tag{A}$$

It is a simple exercise in algebraic manipulation to show that a and b are relatively prime, as well as to demonstrate several interesting divisibility properties of K. If the right and left hand terms of (4) are expanded and appropriately regrouped, the following expression may be obtained:

$$\begin{aligned} K^n=nab\Big\{&\Big[(n-1)K^{n-2}+\frac{(n-1)(n-2)}{2!}K^{n-3}a+\cdots+(n-1)Ka^{n-3}+a^{n-2}\Big]\\ &+\frac{(n-1)}{2!}b\Big[(n-2)K^{n-3}+\frac{(n-2)(n-3)}{2!}K^{n-4}a+\cdots+(n-2)Ka^{n-4}+a^{n-3}\Big]\\ &+\cdots+\frac{(n-1)}{2!}b^{n-3}[2K+a]+b^{n-2}\Big\}\end{aligned} \tag{8}$$

if n is a prime. However, as is well known [1], only prime values of n need be considered for complete generality. From (8) a powerful statement can be made concerning the divisibility of K, namely

$$nab\mid K^n. \tag{B}$$

As a trivial consequence of (B) we have the following two results:

$$\left.\begin{array}{ll} & (K,a)\neq 1 \quad \text{unless } a=1 \\ \text{and similarly} & \\ & (K,b)\neq 1 \quad \text{unless } b=1 \end{array}\right\}. \tag{C}$$

From (B) and (C) we conclude that if $(a, b)=u\neq 1$, then $u\mid K^n$ so that $(K, u)=v\neq 1$, and we can factor at least v from each term of (4) to reduce the equation to an equivalent but lower form, contrary to the assumption that (4) was in its lowest common form. Therefore

$$(a,b)=1. \tag{D}$$

Since n is a prime and from (B) n divides K^n, then necessarily

$$(K,n)=n. \tag{E}$$

In view of the results given in (C), we shall investigate whether $(K, a)=a$ or $(K, b)=b$ is ever possible except when $a=1$ or $b=1$. Assume $(K, a)=a\neq 1$. Then $K=ka$. Substituting into (8) we get

$$k^n a^n = nab\left\{\left[(n-1)k^{n-2}a^{n-2}+\frac{(n-1)(n-2)}{2!}k^{n-3}a^{n-2}+\cdots+(n-1)ka^{n-2}+a^{n-2}\right]\right.$$

$$(9)\qquad +\frac{(n-1)}{2!}b\left[(n-2)k^{n-3}a^{n-3}+\frac{(n-2)(n-3)}{2!}k^{n-4}a^{n-3}\right.$$

$$\left.\left.+\cdots+(n-2)ka^{n-3}+a^{n-3}\right]+\cdots+\frac{(n-1)}{2!}b^{n-3}[2ka+a]+b^{n-2}\right\}.$$

Then a divides every term on the right except b^{n-2} since $(a, b)=1$, which is a contradiction for $n>2$. Therefore

$$\text{(F)}\qquad \left.\begin{array}{r}(K, a) \neq a \\ \text{and similarly if } b \neq 1 \\ (K, b) \neq b\end{array}\right\}.$$

It is difficult to bound the relative values of a, b and K by purely elementary means; however, the following weak bounds are given. Since by (A) $a \neq b$, we can with no loss of generality assume that $b>a$.

If (4) is expanded about the indicated terms

$$(K + (a))^n + (K + (b))^n = (K + (a + b))^n$$

the following expression is obtained:

$$\begin{aligned} K^n &= \frac{n(n-1)}{2!}\left\{(a+b)^2-(a^2+b^2)\right\}+\cdots \\ (10)\qquad &+\frac{n(n-1)}{2!}\left\{(a+b)^{n-2}-(a^{n-2}+b^{n-2})\right\} \\ &\quad n\left\{(a+b)^{n-1}-(a^{n-1}+b^{n-1})\right\} \\ &\quad \left\{(a+b)^n-(a^n+b^n)\right\}. \end{aligned}$$

From (10) it is possible to write the following inequality (which becomes an equality only for $n=2$):

$$K^n - \left\{(a+b)^n - (a^n+b^n)\right\} \geqq 0.$$

If attention is restricted to cases for which $n>2$ and the inequality is enhanced by replacing b by a:

$$K^n - (2^n - 2)a^n > 0, \text{ or}$$

$$\text{(G)}\qquad K > (2^n - 2)^{1/n}a.$$

Similarly, if a is replaced by b in (10) and the resulting inequality is enhanced by adding those terms missing to complete the nth power, the following inequality is obtained:

$$\text{(H)}\qquad K < 2b + 1.$$

The results given in (A) through (H) by no means exhaust the conditions on

(1) obtainable by elementary algebraic operations on (4). In particular, it is possible to derive the Legendre formulae [1, p. 89] by appealing only to arguments of relative primality and divisibility. However, the purpose of this paper is to present the Reis formulation, which was unpublished, and to indicate the fruitfulness of this form when subjected to even the most rudimentary analysis.

We shall close by applying the results already obtained to the case $n=2$, to define an interesting variation on the commonly given general solution of the Pythagorean problem. When $n=2$, (8) becomes $K^2=2ab$. Since $(a, b)=1$, and since K is even

$$K^2 = 2^{2R}m^2n^2 \tag{11}$$

where m and n are odd relatively prime integers. From (11) it is apparent that

$$\begin{aligned} a &= 2^{2R-1}m^2 \\ b &= n^2 \\ K &= (2ab)^{1/2} = 2^R mn. \end{aligned} \tag{12}$$

If these results are substituted into (4), the general expression for Pythagorean triples is

$$\begin{aligned} z &= 2^R mn + 2^{2R-1}m^2 + n^2 \\ &= (2^{R-1}m + n)^2 + (2^{R-1}m)^2 \\ y &= 2^R mn + n^2 \\ &= (2^{R-1}m + n)^2 - (2^{R-1}m)^2 \\ x &= 2^R mn + 2^{2R-1}m^2 \\ &= 2(2^{R-1}m + n)(2^{R-1}m) \end{aligned} \tag{13}$$

which is, of course, identical to the usual solution form for Pythagorean triples

$$\begin{aligned} x &= 2ab \\ y &= a^2 - b^2 \qquad (a, b) = 1 \\ z &= a^2 + b^2 \qquad 2 \mid x. \end{aligned} \tag{14}$$

The general solution given by (13) is an interesting variant of the commonly given form (14). The first few Pythagorean triples as computed using (13) are tabulated below:

m	n	R	x	y	z
1	1	1	3	4	5
1	3	1	8	15	17
1	1	2	5	12	13
3	1	1	7	24	25.

This work was supported by the U. S. Atomic Energy Commission.

Reference

1. R. D. Carmichael, The theory of numbers and diophantine analysis, Dover, New York, reprint of 1914 ed., pp. 87–88.

THE MATHEMATICS OF THE ROUND-ROBIN

HENRY WINTHROP, University of South Florida

Let me recall the nature of a round-robin letter. A sends a letter, containing a given message, to each of m individuals, requesting each of them to note down the message for themselves and to retranscribe it in m separate letters and send each of these m letters on in turn to different friends. Each such retranscription, of course, contains the same request. If the propagation of this message can take place for any finite length of time in a given total population of M potential respondents, the following two questions can arise. 1. What equation will reflect the growth of receivers of the round-robin letter, assuming that no *convergence of transmission* occurs, that is to say, no person in the population ever receives a round-robin letter twice? 2. What equation will reflect the growth, assuming that such convergence does take place? We shall answer these two questions at this point.

Before doing so, however, let us note that in the usual meaning of a round-robin letter, we deal with a one-shot affair, that is to say, a receiver retranscribes the letter m times for each of his m acquaintances, forwards these m transcriptions so that *ideally* they are received in the unit of time following the one in which he, himself, received the round-robin letter and thereafter the receiver never again circulates the round-robin. This situation can be represented by the following tableau in which Δ_i represents the increase in the number of recipients at $t=i$ and in which we adopt the convention that $\Delta_0=1$.

(A)

t	Δ_i
0	1
1	m
2	m^2
3	m^3
.	.
.	.
.	.
i	m^i

so that

$$N(t) = \frac{m(m^t) - 1}{m - 1} \tag{1a}$$

where, in the present context, equation (1a) is the analogue of the standard formula for the sum of a geometric progression, namely,

$$s = \frac{rl - a}{r - 1} \tag{1b}$$

in which $a=1$, $r=m$ and $l=m^i$.

In the present context let us assume that if each receiver possesses *an average number of* acquaintances, D, where D is taken as an integral multiple of m, that he will be sufficiently interested in the task of propagating the round-robin message, to retranscribe the letter over an interval of D/m time units and will forward the letter to *m acquaintances only* per unit time. We again assume that a round-robin letter is in the hands of any receiver one time unit after its transmitter has retranscribed it and that the retranscription takes place during the same unit of time in which it has been received. Under these assumptions the growth of the round-robin will be represented by the following tableau:

$$
\text{(B)} \qquad
\begin{array}{ll}
\underline{t} & \underline{\Delta_i} \\
0 & \Delta_0 = 1 \\
1 & \Delta_1 = m = m(m+1)^0 \\
2 & \Delta_2 = m + m^2 = m(m+1)^1 \\
3 & \Delta_3 = m + 2m^2 + m^3 = m(m+1)^2 \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
t & \Delta_t = m(m+1)^{t-1}.
\end{array}
$$

In dealing with tableaus A and B and equations (1a) and (1b) we have assumed that no duplication of message reception would occur for any receiver, that is to say, senders would not send the round-robin letter to the same persons. Clearly the larger the value of M and the smaller the value of m, the more reasonable will this assumption be. Throughout this paper all calculations for $N(t)$ will then be based upon *these assumptions of nonconvergence*. The situation with respect to *convergence* is explored by means of tableau C and equation (2).

In the light of the preceding paragraph and the structural equations of tableau B, if we now wish to obtain $N(t)$, *the total number of persons who will have received the round-robin letter*, at time t, we have that

$$
\text{(1c)} \qquad N(t) = \sum_{i=0}^{t-1} \Delta_i = \sum_{i=0}^{t-1} m(m+1)^i + 1 = (m+1)^t
$$

which is the exponential organic growth law.

Now let us try to answer the second question which was posed above, which assumes convergence in transmission within a population of M potential respondents.

Clearly convergence cannot occur until $t \geqq 2$. The probability of such convergence will, of course, change with the passage of time. The probability of receiving a round-robin at time, t, for the second time or, for that matter, for the nth time, is $N(t-1)/M$ where $N(t-1)$ represents the total number of persons in the population, M, who have already received it over $[1, (t-1)]$. If we let $m/M = K$, then the growth under convergence is given by the following incomplete tableau:

$$N(0) = 1$$

$$N(1) = (m+1)$$

$$\text{(C)}\quad N(2) = N(1)m\left[\frac{M - N(1)}{M}\right] + N(1) = \Delta_2 + N(1) = (1 - K)[N(1)]^2$$

$$N(3) = (1 - K)[N(1)]^3 - K(1 - K)^2[N(1)]^4$$

$$\vdots$$

It can then be shown that the growth under convergence will be given by the recursion relation

$$N(t) = N(1)[N(t - 1)] - K[N(t - 1)]^2 \tag{2}$$

where, it is to be remembered, $N(1) = m+1$ and $N(t)$ is an *expected value*.

Let us now return to the first condition of nonconvergence, since it offers so many interesting and unexpected consequences under a variety of socially and psychologically realistic constraints. Let us assume once again that each receiver of the round-robin letter was sufficiently interested in it to follow its directions more than once. This means that he writes to a batch of m acquaintances in each of several successive units of time. We must recognize that even the ideal growth involved in nonconvergence, is curtailed by the fact that every transmitter has a limited circle of acquaintances, D, which we shall assume is an average for each member of the population, M. Knowing this average we can correct equation (1c) for this fact. Essentially what is involved here is that each transmitter of the round-robin ceases to be able to contact any friends or acquaintances after a given period of time. Let us assume that this period consists of K units of time. This means that (1) each receiver of the round-robin message contacts m and only m friends per unit time; (2) each receiver makes these contacts in k successive units of time with no gaps or omissions in the time sequence; that is to say, he writes to m friends in each of the next k units of time following the one on which he received the round-robin message, until he has made a total of D contacts, where $D = km$; and (3) after k such units of time have elapsed, each transmitter thereafter ceases to be effective in adding to the growth in numbers of those who have received the round-robin letter. Under these assumptions let us determine $N(t)$ when $t = k+i$ and $i \leqq k$. That is to say, we wish to determine $N(t)$ where $N(k) < N(t) \leqq N(2k)$. We note that for our original conditions governing equation (1c), where $1 \leqq i \leqq k$, $\Delta_i = mY^{i-1}$, where $Y = (m+1)$. At $t = k+1$, the members of Δ_0 will cease to function as transmitting agents so that only $N(k) - \Delta_0$ of potential transmitters can contribute to the growth of the round-robin message. We shall therefore have that

$$N(k + 1) = [N(k) - \Delta_0]m + N(k) \tag{3}$$

$$= N(k) + \Delta_{k+1} \tag{4}$$

$$= N(k)Y - mY^0 \tag{5}$$

$$= N(k)Y - mY^{-1}(Y + 0 \cdot m) \tag{6}$$

where the necessity for the form of equation (6) will soon become apparent. The form of equation (3) consists of two parts: $[N(k) - \Delta_0]m = \Delta_{k+1}$ and $N(k)$, which represents the fact that the total number of receivers of the round-robin message at $t=k+1$ consists of the increase which occurred at this value of t plus the accumulated growth over $[1, k]$. This is, of course, restated by equation (4).

At $t=k+2$, the members of Δ_1 cease to function as transmitting agents, so that we have

$$N(k+2) = [N(k+1) - \Delta_1]m + N(k+1) \tag{7}$$

$$= [N(k+1) - mY^0]m + N(k+1) \tag{8}$$

$$= N(k+1)Y - m^2Y^0. \tag{9}$$

Substituting equation (5) in equation (9) we obtain

$$N(k+2) = [N(k)Y - mY^0]Y - m^2Y^0 \tag{10}$$

$$= N(k)Y^2 - mY^0(Y + 1 \cdot m). \tag{11}$$

In general when $t=k+i$, $i \leqq k$, the members of Δ_{i-1} *drop out* of the effective population of transmitters, becoming *deadwood* so far as transmitting potential is concerned. Therefore changes in $N(t)$ for $(k+i)$ when $(k+1) \leqq t \leqq 2k$, will be a function of the accumulating number of drop outs. We shall accordingly refer to $N(t)$ over $[(k+1), 2k]$ as the *drop out function* for $N(t)$. If the process of iteration employed in order to obtain $N(k+1)$ and $N(k+2)$ is extended in order to obtain $N(k+i)$ we get

$$N(k+i) = N(k)Y^i - mY^{i-2}[Y + (i-1)m] \tag{12}$$

where $i \leqq k$.

Substituting $(m+1) = Y$ in equation (12) we obtain the total number of individuals who have received the round-robin letter under the constraint that every transmitter will exhaust his $D=km$, over k units of time. Making the substitution gives

$$N(k+i) = (m+1)^{k+i} - m(m+1)^{i-1} - (i-1)m^2(m+1)^{i-2} \tag{13}$$

$$= (m+1)^{i-2}[(m+1)^{k+2} - m^2(i-1) - m(m+1)] \tag{14}$$

$$= (m+1)^{i-2}[(m+1)^{k+2} - m(im+1)] \tag{15}$$

which is the appropriate recursion relationship.

Let us now look at the condition of nonconvergence from a different angle. We shall now assume that each transmitter of the round-robin letter writes to a variable number of acquaintances in successive units of time. Originally we assumed that every transmitter met a *constant* number of persons, m, per unit time. Suppose we call the function which reflects the variation in the number of one's acquaintances one contacts over the course of time, $\phi(t)$, leaving the unit of time unspecified. Then in our original case $\phi(t)=m$; that is, $\phi(t)$ is a straight line function for all transmitters. However, let us now reexamine our original set of conditions assuming that $\phi(t) \neq m$ and that $\phi(t)$ is monotone increasing or decreasing. (This assumption is made for simplicity. Actually $\phi(t)$ may be any function whatsoever.) This will mean that *for all transmitters*, the number of contacts made per unit time is the same. In short, when $t=1$, all transmitters make $\phi(1)$ contacts, when $t=2$, $\phi(2)$ contacts, and when $t=n$, $\phi(n)$ contacts. Upon the basis of these additional assumptions our original set of conditions will give rise to the following set of increases:

$$
\text{(D)}\qquad
\begin{array}{ll}
t & \Delta_t \\
0 & \Delta_0 = 1 \\
1 & \Delta_1 = \phi(1) \\
2 & \Delta_2 = \phi(2) + \phi(1)\phi(2) = \phi(2) + \phi(2)\alpha_1^1 \\
3 & \Delta_3 = \phi(3) + \phi(1)\phi(3) + \phi(2)\phi(3) + \phi(1)\phi(2)\phi(3) \\
\cdot & \cdot \;= \phi(3) + \phi(3)\alpha_1^2 + \phi(3)\alpha_2^2 \\
\cdot & \cdot \\
\cdot & \cdot \\
t & \Delta_t = \phi(t) + \phi(t)\alpha_1^{t-1} + \phi(t)\alpha_2^{t-1} + \cdots + \phi(t)\alpha_{t-1}^{t-1}
\end{array}
$$

where $\alpha_1^1=\phi(1)$, $\alpha_1^2=\phi(1)+\phi(2)$, $\alpha_1^3=\phi(1)+\phi(2)+\phi(3)$, $\alpha_2^3=\phi(1)\phi(2)+\phi(2)\phi(3)+\phi(1)\phi(3)$, etc. We would then have for the period of generation, $[1, k]$, the following expression for $N(k)$:

$$
N(k) = \sum_{t=1}^{k} \Delta_t + 1 = \sum_{t=1}^{k} \phi(t) + \sum_{t=2}^{k}\left[\phi(t)\sum_{i=1}^{t-1}\alpha_i^{t-1}\right] + 1. \tag{16}
$$

It is of interest to see what the tableau of these expansions will look like when $\phi(t)$ is specified. Let us therefore assume $\phi(t)=t$, a fairly simple function. We would then obtain the series 1, 4, 18, 96 $\cdots$ in which the general term is given by $\Delta_i=(i+1)!-i!=i(i)!$ so that the expression for $N(t)$ would be given by

$$
N(t) = \sum_{x=1}^{t} x(x)! + 1. \tag{17}
$$

In reality any receiver's interest in spreading the message contained in the round-robin letter, is variable; that is to say, it may wax, wane, or fluctuate, depending upon a set of psychological and social factors which affect everyone the same way. Suppose then that $\phi(t)$ is a *phase function* which measures the *varying intensity of interest* displayed by any transmitter in retranscribing and spreading the message. If this phase function is common to all transmitters then, in a sense, each transmitter operates in a *private time* which is embedded in *public, calendar time.* By private time we mean that in general different transmitters who are at phase, $\phi(i)$, may be at that phase for different values of t and that in general for any given value of t different transmitters may be operating psychologically at different phases of $\phi(t)$. If this is one of the underlying psychological conditions of transmission, then the variable increments of growth of the round-robin will be represented by the following model, where Δ_i represents the increment of transmission at $t=i$:

$$
\text{(E)}\qquad
\begin{aligned}
\Delta_0 &= 1 \\
\Delta_1 &= \phi(1) \\
\Delta_2 &= \phi(2) + \phi^2(1)
\end{aligned}
$$

$$\Delta_3 = \phi(3) + 2\phi(1)\phi(2) + \phi^3(1)$$
$$\vdots$$
$$\Delta_i = F(h_i, \phi)$$

in which $F(h_i, \phi)$ designates the partitions of the integer, i, and in which the coefficient of each partition represents the number of possible permutations of that partition. If, however, we assume specific expressions for $\phi(t)$, this model will generate some well-known series so that the absence of any general difference equation expression works no hardship. If, for instance, we assume $\phi(t) = t$, the model will generate the *alternate* terms of the Fibonacci series $1+2+3+5+8+13+21 \cdots$.

If we assume $\phi(t) = kt$, the model which will then be generated for the schema represented by tableau (E) is the following:

$$\text{(F)} \quad \begin{aligned} \Delta_0 &= 1 \\ \Delta_1 &= k \\ \Delta_2 &= 2k + k^2 \\ \Delta_3 &= 3k + 4k^2 + k^3 \\ \Delta_4 &= 4k + 10k^2 + 6k^3 + k^4 \\ \Delta_5 &= 5k + 20k^2 + 21k^3 + 8k^4 + k^5 \\ &\vdots \\ \Delta_i &= ik + \frac{[(i-1)+2]!k^2}{(i-2)!3!} + \frac{[(i-2)+4]!k^3}{(i-3)!5!} + \frac{[(i-3)+6]!k^4}{(i-4)!7!} \\ &\quad + \cdots + \frac{[i-(n-1)+2(n-1)]!k^n}{(i-n)!(2n-1)!} + \cdots + \frac{k^i}{(i-n)!(2n-1)!} \end{aligned}$$

where the coefficient of k^n is the nth term of the $2r$th order of the figurate numbers and where the nth term of the rth order of figurate numbers is given by

$$\frac{(n+r-2)!}{(n-1)!(r-1)!}.$$

In terms of i, however, the coefficient of k^n is the $[i-(n-1)]$-th term of the $2n$th order of the figurate numbers.

From the preceding it should also be clear that $N(t)$ is a polynomial of the following form:

$$(18) \quad N(t) = k\overset{2}{S}_t + \overset{2}{k}\overset{4}{S}_{t-1} + \overset{3}{k}\overset{6}{S}_{t-2} + \cdots + \overset{t}{k}\overset{2r}{S}_{t-(r-1)} + 1, \qquad r = 1, 2, \cdots n.$$

$N(t)$ may be evaluated by noting that the sum of n terms of the rth order is $n(n+1)(n+2) \cdots (n+r-1)/r!$ which is the nth term of the $(r+1)$-th order. In equation (18) above $S^{2r}{}_{t-(r-1)}$ is the sum of the first $t-(r-1)$ terms of the $2r$th order of the figurate numbers. This expression for $N(t)$ is, of course, obtained by aligning all the polynomials of the form represented by tableau (F) and adding our results vertically.

All of the preceding results will, I believe, make the reader realize the variety of growth structures obtainable with the round-robin under different possible assumptions. All of the series obtained in the present paper are divergent and, if translated into functions of a continuous variable, namely, in this case, t, they will prove to be exponential in every case. The discrete values of these finite difference equations can be used as points through which the proper exponential can be obtained by standard, curve-fitting methods.

Diffusion equations—for these are what the models discussed in this paper are essentially—have been the focus of attention for several mathematicians and social scientists. Among these we may note the following. A brief but clear account of several diffusion models has been furnished by Karlsson [1]. A paper by Rapoport and Rebhun [2] represents a technical and detailed approach to the subject on the part of two mathematicians. An elaborate account of *imitative behavior*, which involves a great deal of diffusion analysis, occurs in a fairly well-known volume by Rashevsky [3]. Two contributions in this direction were a pioneer paper and later an extended treatment of the subject in a doctoral dissertation, by Winthrop [4, 5]. There is, of course, a richer vein of literature along these lines than has been suggested here, but the reader will have little difficulty in locating it if he is interested in the subject.

References

1. G. Karlsson, Social mechanisms, Studies in Sociological Theory, The Free Press, Glencoe, Illinois, 1958.

2. A. Rapoport and L. I. Rebhun, On the mathematical theory of rumor spread, Bull. Math. Biophys., 14 (1952) 375–383.

3. N. Rashevsky, Mathematical biology of social behavior, University of Chicago Press, Chicago, 1951.

4. H. Winthrop, A kinetic theory of socio-psychological diffusion, Journal of Social Psychology, 22 (1945) 31–60.

5. ———, A theory of behavioral diffusion, A contribution to the mathematical biology of social phenomena. Unpublished thesis submitted to the Faculty of the New School for Social Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy, 1953.

ON THE SOLUTION OF THE REAL QUARTIC

WILLIAM F. CARPENTER, Central Florida Junior College

1. Introduction. One method of solving a real quartic requires the solution of a resolvent cubic, from which two real quadratic equations are formed. The solutions to these quadratic equations are the solutions to the quartic. However, not every root of the resolvent cubic will yield a solution to the quartic if all of the roots of the quartic are complex.

This paper presents an approach to the solution of the quartic which leads to a cubic whose positive roots always produce a solution to the quartic, even aligning all the polynomials of the form represented by tableau (F) and adding our results vertically.

All of the preceding results will, I believe, make the reader realize the variety of growth structures obtainable with the round-robin under different possible assumptions. All of the series obtained in the present paper are divergent and, if translated into functions of a continuous variable, namely, in this case, t, they will prove to be exponential in every case. The discrete values of these finite difference equations can be used as points through which the proper exponential can be obtained by standard, curve-fitting methods.

Diffusion equations—for these are what the models discussed in this paper are essentially—have been the focus of attention for several mathematicians and social scientists. Among these we may note the following. A brief but clear account of several diffusion models has been furnished by Karlsson [1]. A paper by Rapoport and Rebhun [2] represents a technical and detailed approach to the subject on the part of two mathematicians. An elaborate account of *imitative behavior*, which involves a great deal of diffusion analysis, occurs in a fairly well-known volume by Rashevsky [3]. Two contributions in this direction were a pioneer paper and later an extended treatment of the subject in a doctoral dissertation, by Winthrop [4, 5]. There is, of course, a richer vein of literature along these lines than has been suggested here, but the reader will have little difficulty in locating it if he is interested in the subject.

References

1. G. Karlsson, Social mechanisms, Studies in Sociological Theory, The Free Press, Glencoe, Illinois, 1958.

2. A. Rapoport and L. I. Rebhun, On the mathematical theory of rumor spread, Bull. Math. Biophys., 14 (1952) 375–383.

3. N. Rashevsky, Mathematical biology of social behavior, University of Chicago Press, Chicago, 1951.

4. H. Winthrop, A kinetic theory of socio-psychological diffusion, Journal of Social Psychology, 22 (1945) 31–60.

5. ———, A theory of behavioral diffusion, A contribution to the mathematical biology of social phenomena. Unpublished thesis submitted to the Faculty of the New School for Social Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy, 1953.

ON THE SOLUTION OF THE REAL QUARTIC

WILLIAM F. CARPENTER, Central Florida Junior College

1. Introduction. One method of solving a real quartic requires the solution of a resolvent cubic, from which two real quadratic equations are formed. The solutions to these quadratic equations are the solutions to the quartic. However, not every root of the resolvent cubic will yield a solution to the quartic if all of the roots of the quartic are complex.

This paper presents an approach to the solution of the quartic which leads to a cubic whose positive roots always produce a solution to the quartic, even in the case where the roots of the quartic are all complex. Furthermore, the solution to the quartic can be written directly in terms of its coefficients without having to solve any intermediate quadratic equations.

2. The solution of $x^4+Cx^2+Dx+E=0$. Let us form the cubic

$$y^3 + 2Cy^2 + (C^2 - 4E)y - D^2 = 0.$$

Suppose r is one of the positive roots of this cubic; then the roots of the quartic are given by

$$x_{1,2} = \frac{\sqrt{r}}{2} \pm \sqrt{\left\{-\frac{r}{4} - \frac{C}{2} - \frac{D}{2\sqrt{r}}\right\}}$$

$$x_{3,4} = \frac{-\sqrt{r}}{2} \pm \sqrt{\left\{-\frac{r}{4} - \frac{C}{2} + \frac{D}{2\sqrt{r}}\right\}}.$$

3. Justification. Consider the roots of the quartic to be made up in conjugate pairs as follows: $x_{1,2}=\alpha\pm\beta$; $x_{3,4}=\delta\pm\gamma$. We can now write $\{(x-\alpha)^2-\beta^2\}\cdot\{(x-\delta)^2-\gamma^2\}=0$. From this we get

$$x^4 - 2(\alpha + \delta)x^3 + \{(\alpha + \delta)^2 + 2\alpha\delta - \beta^2 - \gamma^2\}x^2 - \{2\alpha\delta(\alpha + \delta) - 2\delta\beta^2 - 2\alpha\gamma^2\}x + \{\alpha^2\delta^2 - \beta^2\delta^2 - \alpha^2\gamma^2 + \beta^2\gamma^2\} = 0.$$

Equating coefficients with $x^4+Cx^2+Dx+E=0$ and noting that the coefficient of the x^3 term is zero, we get three simultaneous equations

(1) $$-2\alpha^2 - \beta^2 - \gamma^2 = C$$

(2) $$-2\alpha\beta^2 + 2\alpha\gamma^2 = D$$

(3) $$\alpha^4 - \beta^2\alpha^2 - \alpha^2\gamma^2 + \beta^2\gamma^2 = E.$$

Solving these equations for α we have $64\alpha^6+32C\alpha^4+(4C^2-16E)\alpha^2-D^2=0$. Letting $y=4\alpha^2$ we get

(4) $$y^3 + 2Cy^2 + (C^2 - 4E)y - D^2 = 0.$$

Now for $D\neq 0$ equation (4) will have at least one positive root since its constant term will always be negative. Suppose r is one of its positive roots then $\alpha=\sqrt{r}/2$. Substituting this value of α into equations (1) and (2) we find that $\beta^2=-r/4-C/2-D/2\sqrt{r}$ and $\gamma^2=-r/4-C/2+D/2\sqrt{r}$. The roots to the quartic are thus

(5) $$x_{1,2} = \alpha \pm \beta = \frac{\sqrt{r}}{2} \pm \sqrt{\left\{-\frac{r}{4} - \frac{C}{2} - \frac{D}{2\sqrt{r}}\right\}}$$

and since $\delta=-\alpha$

(6) $$x_{3,4} = -\alpha \pm \gamma = -\frac{\sqrt{r}}{2} \pm \sqrt{\left\{-\frac{r}{4} - \frac{C}{2} + \frac{D}{2\sqrt{r}}\right\}}.$$

4. Sample problem. Let us find the roots of

$$x^4 + 2x^2 + 32x + 65 = 0.$$

Here $C=2$, $D=32$, $E=65$. Forming the cubic

$$y^3 + 2Cy^2 + (C^2 - 4E)y - D^2 = 0$$

we get $y^3+4y^2-256y-1024=0$. Solution of this cubic yields $y=16$ as one of its positive roots. Hence the roots of the given quartic are by equations (5) and (6)

$$x_{1,2} = \frac{\sqrt{16}}{2} \pm \sqrt{\left\{-\frac{16}{4} - \frac{2}{2} - \frac{32}{2\sqrt{16}}\right\}}$$

and

$$x_{3,4} = -\frac{\sqrt{16}}{2} \pm \sqrt{\left\{-\frac{16}{4} - \frac{2}{2} + \frac{32}{2\sqrt{16}}\right\}}$$

which reduce to $x_{1,2}=2\pm 3i$ and $x_{3,4}=-2\pm i$.

A DECOMPOSITION OF THE INTEGERS TO GENERATE GRAPHS

N. R. DILLEY, Marina High School, Huntington Beach, California

Claude Berge [1, p. 5] introduces a graph as the situation when one has (a) a set X and (b) a function Γ mapping X into X.

The decomposition of the integers, under a given set of base integers, reveals structures which, when projected to a point space or set space, determine graphs. One observes that in this graph space any two points (x, y) are so related that one may draw an oriented line from one point, x, to the other, y. A given point x may be a vertex, and a pair of points x, y, is an arc of the graph. A path is a sequence $(v_1, v_2, \cdots)$ of arcs where each arc is distinct and precedes another (except for the terminal arc). A path is simple if it does not use the same arc twice (no loops present). The length of a path is the number of arcs in the sequence. A path may be finite or infinite.

This paper is an elementary discussion of the proposition that any integer greater than 1, under certain conditions, has a realization as a single graph. The argument for the proposition will be inductive.

Let the consecutive positive integers be listed as the array:

$$(1)\qquad \begin{matrix} C = 1 & 2 & \cdots k \\ 2 & 3 & \cdots 2k \\ 3 & 4 & \cdots 3k \\ \cdot & \cdot & \cdots \cdot \\ n & n+1 & \cdots nk. \end{matrix}$$

4. Sample problem. Let us find the roots of

$$x^4 + 2x^2 + 32x + 65 = 0.$$

Here $C=2$, $D=32$, $E=65$. Forming the cubic

$$y^3 + 2Cy^2 + (C^2 - 4E)y - D^2 = 0$$

we get $y^3+4y^2-256y-1024=0$. Solution of this cubic yields $y=16$ as one of its positive roots. Hence the roots of the given quartic are by equations (5) and (6)

$$x_{1,2} = \frac{\sqrt{16}}{2} \pm \sqrt{\left\{-\frac{16}{4} - \frac{2}{2} - \frac{32}{2\sqrt{16}}\right\}}$$

and

$$x_{3,4} = -\frac{\sqrt{16}}{2} \pm \sqrt{\left\{-\frac{16}{4} - \frac{2}{2} + \frac{32}{2\sqrt{16}}\right\}}$$

which reduce to $x_{1,2}=2\pm 3i$ and $x_{3,4}=-2\pm i$.

A DECOMPOSITION OF THE INTEGERS TO GENERATE GRAPHS

N. R. DILLEY, Marina High School, Huntington Beach, California

Claude Berge [1, p. 5] introduces a graph as the situation when one has (a) a set X and (b) a function Γ mapping X into X.

The decomposition of the integers, under a given set of base integers, reveals structures which, when projected to a point space or set space, determine graphs. One observes that in this graph space any two points (x, y) are so related that one may draw an oriented line from one point, x, to the other, y. A given point x may be a vertex, and a pair of points x, y, is an arc of the graph. A path is a sequence $(v_1, v_2, \cdots)$ of arcs where each arc is distinct and precedes another (except for the terminal arc). A path is simple if it does not use the same arc twice (no loops present). The length of a path is the number of arcs in the sequence. A path may be finite or infinite.

This paper is an elementary discussion of the proposition that any integer greater than 1, under certain conditions, has a realization as a single graph. The argument for the proposition will be inductive.

Let the consecutive positive integers be listed as the array:

$$\text{(1)}\qquad \begin{array}{rcccc} C = & 1 & 2 & \cdots & k \\ & 2 & 3 & \cdots & 2k \\ & 3 & 4 & \cdots & 3k \\ & \cdot & \cdot & \cdots & \cdot \\ & n & n+1 & \cdots & nk. \end{array}$$

The initial row is called the base set or the primitive set of the integer array. If the base set is stated, the entries in successive rows become sequences terminating in $2k, 3k, \cdots, nk$. When each row is considered as a set the organization of the array becomes

$$C = C_1 \cup C_2 \cup C_3 \cup \cdots \cup C_n. \tag{2}$$

One notes that

$$X \to C. \tag{3}$$

Next one may establish an image set to C or a succeeding set to C by describing the function Γ. The function Γ is a transformation T made up of a triplet of operations $\{[a], [b], [c]\}$. The operation $[a]$ is a partition, in all possible ways, of each entry of C into the base integers of the unique C array. The number of elements in the partition is equal to the number of the row from which the C array entry is obtained. The operation $[b]$ is a permutation of the elements in each partition. Lastly, the operation $[c]$ is an addition of the permutations. The result is the image set system. Let this system be called the system of K sets. Stating this quite briefly one has the symbolism:

$$T \cdot C = \{[a], [b], [c]\} \cdot C \to K. \tag{4}$$

Again one notes that

$$X \to X \leftrightarrow C \to K. \tag{5}$$

The transformation T is the resultant of the three noncommutative operations $[a]$, $[b]$, $[c]$.

The K sets may be shown in a general fashion by the following array:

$$\begin{array}{llllll} K = 1 & 1 & \cdots & 1 & 1 & (k \text{ terms}) \\ 1 & 2 & \cdots & 2 & 1 & (2k \text{ terms}) \\ 1 & 3 & \cdots & 3 & 1 & (3k \text{ terms}) \\ \cdot & \cdot & \cdot\ \cdot\ \cdot & \cdot & \cdot & \\ 1 & n & \cdots & n & 1 & (nk \text{ terms}). \end{array} \tag{6}$$

The system K, if an independent array, would be correctly considered a system of sequences. Here, however, due to the existence of the isomorphism that exists between C and K, and the fact that C is a system of sets of consecutive integers, one may allow C to induce a set structure into K. This permits the identification of K as a system of sets.

To show some details of the process that forms elements in the K sets, an example will be given. Let the unique $C_{n,k}$ array be the following:

$$\begin{array}{l} C_{4,5} = 1\ 2\ 3\ 4\ 5 \\ \phantom{C_{4,5} = }2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10 \\ \phantom{C_{4,5} = }3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15 \\ \phantom{C_{4,5} = }4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ \underline{12}\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20 \end{array} \tag{7}$$

Here n is 4 so that we have four rows. Since k is 5 the base set is 1, 2, 3, 4, 5. Let us examine a typical entry, the underlined 12 in the 4th row, to show the projection to the K set. The eight partitions of this number 12 into the 4 elements per partition (since $n=4$) are (5, 5, 1, 1), (4, 4, 2, 2), (5, 3, 2, 2), (4, 5, 2, 1), (3, 3, 3, 3), (3, 3, 2, 4), (3, 1, 4, 4), and, lastly, (5, 3, 3, 1). Let these partitions be, respectively, $p_1, p_2, \cdots, p_8$. Now one finds the permutations possible with these partitions. This is the operation [a] followed by the operation [b]. The permutations are, respectively, (6), (6), (12), (24), (1), (12), (12), and (12). The operation [c] or the addition of the permutations gives the K set entry of 85 as the corresponding entry for the C set entry of the 12.

All the entries for the K sets may be found in this manner, difficult as it is to perform. Fortunately, there is a quicker way. Carlitz [2] suggests a recurrence statement. If $a_{n,r}$ is a K set entry in the nth row and rth column, one may give the recurrence relation for the $k=5$ sets as

$$a_{n+1,r} = a_{n,r-4} + a_{n,r-3} + a_{n,r-2} + a_{n,r-1} + a_{n,r}. \tag{8}$$

This explains that one adds 1, 2, 3, 4, but not more than 5 of the entries above and to the left of the new K set entry. The initial row for any K set is $1 \ 1 \cdots 1$ (k terms).

The array for the $K_{4,5}$ entries would then be:

$$\begin{array}{lllllllllllllllll} K_{4,5} = 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 6 & 10 & 15 & 18 & 19 & 18 & 15 & 10 & 6 & 3 & 1 \\ 1 & 4 & 10 & 20 & 35 & 52 & 58 & 80 & 85 & 80 & 58 & 52 & 35 & 20 & 10 & 4 & 1 \end{array} \tag{9}$$

The general result is that one may associate with any positive integer a triplet of operations that will generate a new number, when n of the primitive set of integers are utilized in the partition. The operations have been stated.

Further examples and discussion of this process which connects the C sets and K sets may be found in the articles by Dilley [**3**].

Now if attention is granted to the structures intermediate between the C sets and the K sets, an extension that produces graphs is of interest. Suppose an example that begins with the C set array of $C_{3,3}$ be used. This is,

$$\begin{array}{lllllll} C_{3,3} = 1 & 2 & 3 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9. \end{array} \tag{10}$$

Next,

$$T(C_{3,3}) \to \{[a] \quad [b] \quad [c]\} C_{3,3} \to K_{3,3}. \tag{11}$$

Therefore,

$$\begin{array}{lllllll} K_{3,3} = 1 & 1 & 1 \\ 1 & 2 & 3 & 2 & 1 \\ 1 & 3 & 6 & \underline{7} & 6 & 3 & 1 \end{array} \tag{12}$$

where the recurrence statement was,

$$a_{n+1,r} = a_{n,r-2} + a_{n,r-1} + a_{n,r}. \tag{13}$$

Before discussing the introduction of graphs it is desirable to state some additional properties of graphs. A graph may be thought of as a set of vertices X, and a function Γ. A graph also may be a set of arcs which determine the function Γ. If the set of arcs is U, then a graph may be listed as $G=(X, U)$, or as $G=(X, \Gamma)$. When arcs are used in the graph going from vertex to vertex, it is customary to show an orientation by the use of an arrowhead on each arc. However, if it is preferred to omit the orientation, one may draw straight lines which are now called edges (not arcs). Again, if the arcs make up a complete circuit (the terminal vertex connects to the initial vertex), the same graph, drawn with the straight lines or edges, is said to present a cycle. A sequence of edges $(u_1, u_2, \cdots)$ determines a simple chain (when the edges attach to the vertices) so that x_i precedes only x_j (excluding the terminal vertex).

Now if one permits a transformation T_1 where T_1 is composed of the two operations [a], a partition, and [b], a permutation of the partition or partitions, then one may realize the result of T_1C as sets of vertices which permit graphs. Hence one may state that

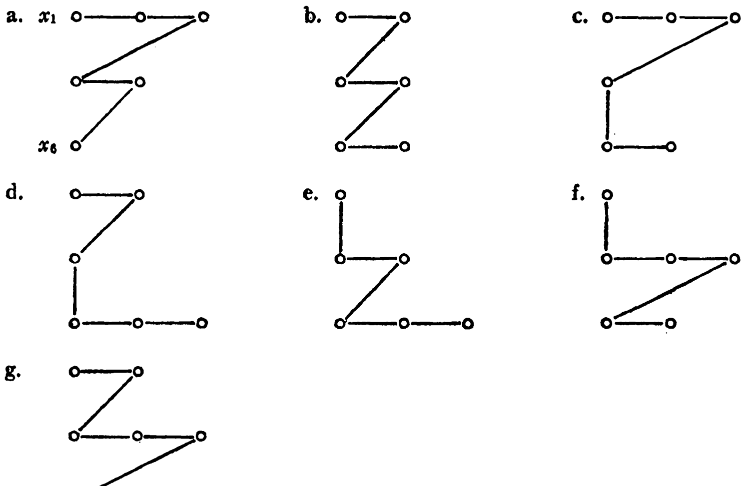
$$T_1C \to G(X, U). \tag{14}$$

To show this for a C set entry, suppose one selects the 6 in the $C_{3,3}$ array, where $n=3$ and $k=3$. The partition and permutation are represented by the statement,

$$T_1C_{3,3} \to G_{3,3} \to (X_{1,2,\dots,7}) \cap (U_{1,2,\dots,7}). \tag{15}$$

The seven possible sets of vertices are shown here with the connecting edges.

(16)



The incidence matrix for any of the seven sets of vertices and connecting edges may be given as

$$A = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{Vmatrix} \tag{17}$$

If $A=(x_{ij})$ then $x_{ij}=1$ when an edge terminates at the x_{ij} vertex, and $x_{ij}=0$ otherwise. The 1 at the x_{12} position indicates that an edge starts from the x_1 vertex and terminates at the x_2 vertex.

Since any one of the seven sets of vertices of $T_1C_{3,3}$ gives the same associated incidence matrix, this suggests that the seven sets of vertices are equivalent to one and only one graph. In general, this must be true, since the graph-space, as a set space, is topological or nonmetrical. This following result is evident:

THEOREM 1. *Any positive integer*, $\neq 1$, *projected to a graph space, equals unity.*

This theorem may be symbolized by the statement,

$$T_1C \rightarrow \{[a], [b]\}C \rightarrow G(U) = 1. \tag{18}$$

The theorem states that any positive integer, associated with a partition followed by a permutation of the partition or partitions, provides sets of vertices. These vertices become points in a graph-space. However, when edges are drawn to connect the vertices in each set (in a consecutive fashion), all of the configurations realize one and the same simple chain graph.

It is instructive to inquire: if a matrix is a realization for a graph, is the reverse also true?

When one proceeds just slightly up from the simple chain in the hierarchy of graphs, one may come to the s-graph [1, p. 27]. This is the graph drawn so that the maximum number of edges between the same two initial and terminal vertices is the number s. As an example one may consider an instructor and his class of students with the relation "x handed in homework to y and y returned same to x." This suggests a 2-graph of order $(n+1)$ where n is the number of students (the order being the total number of vertices in the graph).

In any graph, an oriented edge becomes an arc. A complete circle of arcs makes up a circuit (recall that a circle in graph-space is not the continuous curve of Euclidean 2-space). If an arc terminates at a vertex and then an arc returns to the same initial vertex, then the vertices are said to be of degree 2. If this holds over the entire graph one has a symmetric graph of degree 2 vertices. The extension for degree 3, 4, $\cdots$, n, with partial graphs or factors [1, p. 187] a resultant consequence, readily follows.

It is interesting to note that geometrical structures may lose their identity when projected to a graph-space. A square, for instance, projected to a graph-space may appear as a square, a rhombus, a rectangle, a quadrilateral, or even as a straight line.

References

1. Claude Berge, The theory of graphs and its applications, Wiley, New York, 1962.
2. L. Carlitz, Comment on the paper "Some probability distributions and their associated structures," this MAGAZINE, 37 (1964) 51.
3. N. R. Dilley, Some probability distributions and their associated structures, this MAGAZINE 36 (1963) 175–179; 227–231.

VON AUBEL'S QUADRILATERAL THEOREM

PAUL J. KELLY, University of California, Santa Barbara

The simplest form of Von Aubel's theorem states that if $ABCD$ is a convex quadrilateral, and if squares are erected outwardly on the sides, then the segment joining the centers of one pair of opposite squares is congruent to that joining the centers of the other pair, and the lines of these segments are perpendicular (Fig. 1). One can observe that the center, V_1, of the square on side $\overline{AB}$ can be obtained from the ordered pair (A, B) by starting from the first point A, going half way to B, making a 90° turn in a definite sense, and going the same distance, half the length of $\overline{AB}$, to arrive at V_1 (Fig. 2). If one now gives a circuit order to any four coplanar points A, B, C, D, say the order (A, C, D, B, A), and constructs V_1, V_2, V_3 and V_4 from the successive ordered pairs (A, C), (C, D), (D, B), (B, A), then either $V_1 = V_3$ and $V_2 = V_4$ or else $\overline{V_1V_3} \cong \overline{V_2V_4}$ and the lines of $\overline{V_1V_3}$ and $\overline{V_2V_4}$ are perpendicular. It is no longer necessary that the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ form a convex quadrilateral. The theorem is correct

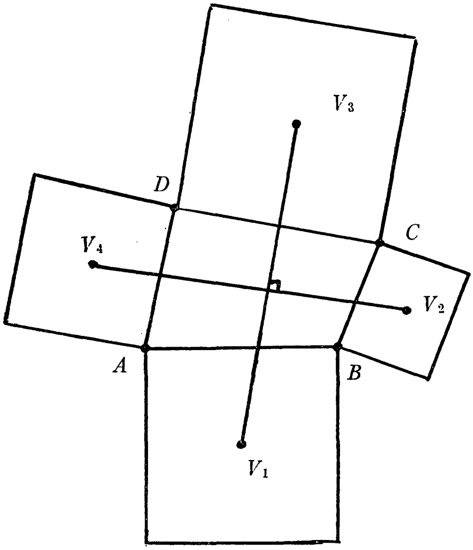


FIG. 1.

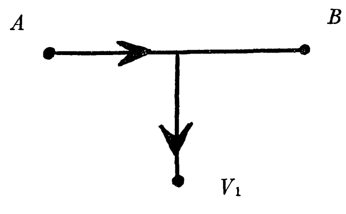


FIG. 2.

References

1. Claude Berge, The theory of graphs and its applications, Wiley, New York, 1962.
2. L. Carlitz, Comment on the paper "Some probability distributions and their associated structures," this MAGAZINE, 37 (1964) 51.
3. N. R. Dilley, Some probability distributions and their associated structures, this MAGAZINE 36 (1963) 175–179; 227–231.

VON AUBEL'S QUADRILATERAL THEOREM

PAUL J. KELLY, University of California, Santa Barbara

The simplest form of Von Aubel's theorem states that if $ABCD$ is a convex quadrilateral, and if squares are erected outwardly on the sides, then the segment joining the centers of one pair of opposite squares is congruent to that joining the centers of the other pair, and the lines of these segments are perpendicular (Fig. 1). One can observe that the center, V_1, of the square on side $\overline{AB}$ can be obtained from the ordered pair (A, B) by starting from the first point A, going half way to B, making a 90° turn in a definite sense, and going the same distance, half the length of $\overline{AB}$, to arrive at V_1 (Fig. 2). If one now gives a circuit order to any four coplanar points A, B, C, D, say the order (A, C, D, B, A), and constructs V_1, V_2, V_3 and V_4 from the successive ordered pairs (A, C), (C, D), (D, B), (B, A), then either $V_1 = V_3$ and $V_2 = V_4$ or else $\overline{V_1V_3} \cong \overline{V_2V_4}$ and the lines of $\overline{V_1V_3}$ and $\overline{V_2V_4}$ are perpendicular. It is no longer necessary that the segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ form a convex quadrilateral. The theorem is correct

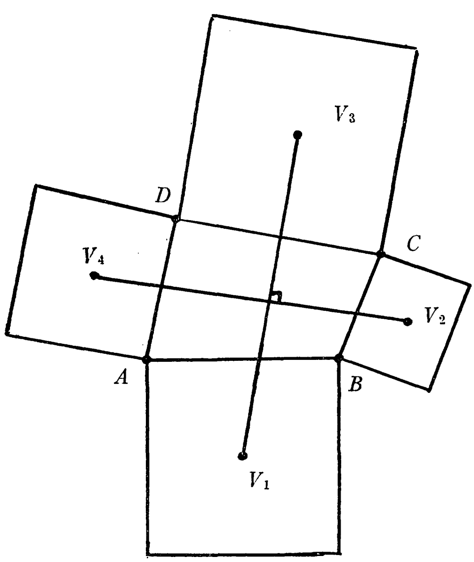


FIG. 1.

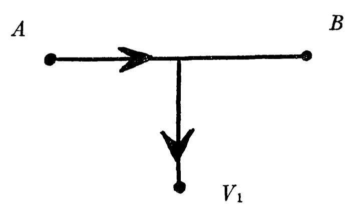


FIG. 2.

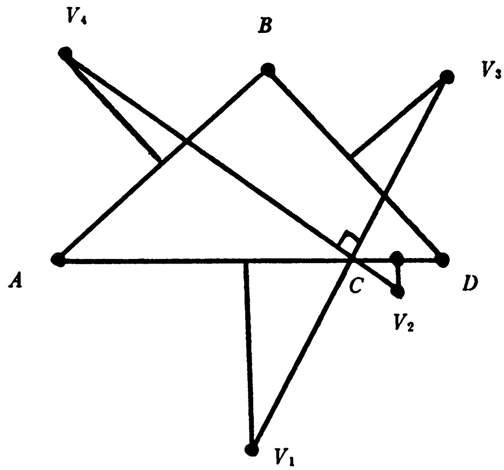


Fig. 3.

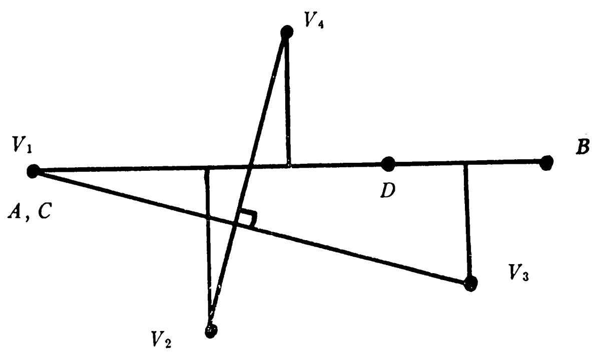


Fig. 4.

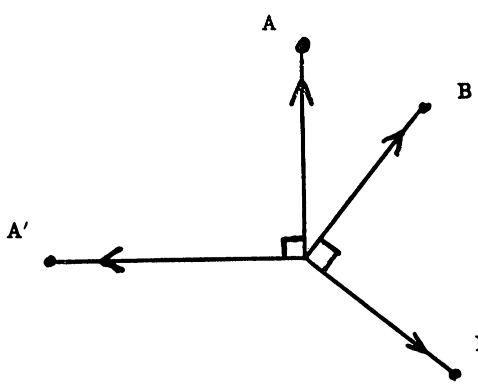


Fig. 5.

even if three, or all four, of the points are collinear (Fig. 3). One can even allow two or more of the points to coincide if the derived point of (A, C), for $C=A$, is also A (Fig. 4). It is necessary, however, to use the same sense of rotation throughout the construction.

The attempt to generalize the Von Aubel property to four noncoplanar points led the writer to a simple space construction and to an easy vector proof for the Von Aubel property in general.

LEMMA. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *denote two vectors at point* P *in a plane. Let* σ *be a rotation of the plane about* P *through* $90°$ *and let* σ^{-1} *be the inverse rotation through* $90°$ *in the opposite sense. If* σ *rotates* $\mathbf{A}$ *to* $\mathbf{A}'$ *and* σ^{-1} *rotates* $\mathbf{B}$ *to* $\mathbf{B}'$ *then the vectors* $\mathbf{A}+\mathbf{B}'$ *and* $\mathbf{B}+\mathbf{A}'$ *are of equal magnitude and are perpendicular* (Fig. 5).

Proof. One can verify this by elementary geometry. Most simply $(\mathbf{A}+\mathbf{B}')\sigma=\mathbf{A}\sigma+(\mathbf{B}\sigma^{-1})\sigma=\mathbf{A}'+\mathbf{B}$ and since σ is a $90°$ rotation, the lemma is established.

Now let $(P_1, P_2, P_3, P_4, P_1)$ denote a 4-circuit in which the points are the vertices of a tetrahedron. Let M_1, M_2, M_3, M_4 be the midpoints respectively of the successive segments $\overline{P_1P_2}, \overline{P_2P_3}, \overline{P_3P_4}, \overline{P_4P_1}$. Then $M_1M_2M_3M_4$ is a parallelogram in what can be called the circuit's midplane. From some arbitrary origin O, let $\mathbf{P}_i$ and $\mathbf{M}_i$ denote the vectors from O to P_i and O to M_i respectively, $i=1, 2, 3, 4$, and let $\mathbf{N}$ denote a unit vector normal to the midplane. Define $\mathbf{V}_1=\mathbf{M}_1+\frac{1}{2}(\mathbf{P}_2-\mathbf{P}_1)\times\mathbf{N}$, $\mathbf{V}_2=\mathbf{M}_2+\frac{1}{2}(\mathbf{P}_3-\mathbf{P}_2)\times\mathbf{N}$, and $\mathbf{V}_3=\mathbf{M}_3+\frac{1}{2}(\mathbf{P}_4-\mathbf{P}_3)\times\mathbf{N}$, $\mathbf{V}_4=\mathbf{M}_4+\frac{1}{2}(\mathbf{P}_1-\mathbf{P}_4)\times\mathbf{N}$. Then the points V_1, V_2, V_3, V_4 lie in the midplane and have the Von Aubel property that $\overline{V_1V_3}\cong\overline{V_2V_4}$ and that the lines of the segments are perpendicular.

To establish the last statement, one need only calculate that $\mathbf{V}_3-\mathbf{V}_1=\mathbf{M}_3-\mathbf{M}_1+(\mathbf{M}_4-\mathbf{M}_2)\times\mathbf{N}$ and $\mathbf{V}_4-\mathbf{V}_2=\mathbf{M}_4-\mathbf{M}_2+(\mathbf{M}_3-\mathbf{M}_1)\times(-\mathbf{N})$. Thus if one defines $\mathbf{A}=\mathbf{M}_3-\mathbf{M}_1$ and $\mathbf{B}=\mathbf{M}_4-\mathbf{M}_2$ to be the parallelogram's diagonal vectors, then $\mathbf{V}_3-\mathbf{V}_1=\mathbf{A}+\mathbf{B}\times\mathbf{N}$ and $\mathbf{V}_4-\mathbf{V}_2=\mathbf{B}+\mathbf{A}\times(-\mathbf{N})$. But if $\mathbf{W}$ is any vector in the midplane, then $\mathbf{W}\times\mathbf{N}$ is the image of $\mathbf{W}$ in a $90°$ rotation of the plane onto itself and $\mathbf{W}\times(-\mathbf{N})$ is the image of $\mathbf{W}$ in the inverse rotation. Hence, by the lemma, $\mathbf{V}_3-\mathbf{V}_1$ and $\mathbf{V}_4-\mathbf{V}_2$ are perpendicular vectors of the same magnitude.

If the points P_i are coplanar (possibly collinear) then the points M_i lie in this plane. If we take $\mathbf{N}$ as a unit normal to the plane, the vector definition of the points V_i yields the usual Von Aubel points and the vector proof is unchanged. The coalescing of two points, for instance $P_1=P_2$, corresponds to defining $M_1=P_1$. Then $\mathbf{P}_2-\mathbf{P}_1$ is the null vector, so $V_1=P_1$. The case $V_3=V_1$ and $V_2=V_4$ occurs when $\mathbf{M}_3-\mathbf{M}_1$ and $\mathbf{M}_4-\mathbf{M}_2$ are perpendicular, are of equal magnitude, and $(\mathbf{M}_4-\mathbf{M}_2)\times\mathbf{N}$ is the rotation of $\mathbf{M}_4-\mathbf{M}_2$ to the position $-(\mathbf{M}_3-\mathbf{M}_1)$. Thus if the midpoint parallelogram is a square, one of the two opposite ways of traversing the circuit, for fixed $\mathbf{N}$, will yield a degenerate derived set in which $V_1=V_3$ and $V_2=V_4$.

ON FINITE SUBSETS OF THE PLANE AND SIMPLE CLOSED POLYGONAL PATHS

MICHAEL GEMIGNANI, University of Notre Dame

Let S be any finite subset of the plane R^2.

I. Is S the set of vertices for at least one simple closed polygonal path (s.c.p.p.)?

II. For how many distinct s.c.p. paths is S the set of vertices?

The primary purpose of this paper is to answer I; however, it will readily be seen that the methods used herein can also be used to answer II as well if one has the patience to sift through the many different cases that can arise.

Set $p(S)$ = number of distinct s.c.p. paths (disregarding orientation) for which S is the set of vertices. If all the points of S are collinear, then (assuming a single point does not form a s.c.p.p.) it follows that $p(S)=0$.

Proposition 1. If all the points of S are not collinear, then $p(S)\geqq 1$.

For any two distinct points x and y of R^2, $\overline{xy}$ will denote the unoriented segment joining x and y, and $f(x, y)$ will denote the straight line determined by x and y. Let $C=[x_0, x_1, \cdots, x_{n-1}, x_n=x_0]$ be a s.c.p.p. in R^2 composed of segments of the form $\overline{x_{i-1}x_i}$, $i=1, \cdots, n$. By the Jordan curve theorem C disconnects R^2 into two open components, one bounded which we denote by $b(C)$, and the other unbounded $u(C)$, the common frontier of both being C.

A line f is called a supporting line of C if $C-f$ is contained in only one component of R^2-f. C will be called convex if $b(C)$ is convex.

Unless specified otherwise, all statements below will refer to C given above.

LEMMA 1. *The following statements are equivalent:*

(a) *C is convex;*

(b) *$b(C)$ is convex;*

(c) $\mathrm{Cl}\, b(C)=b(C)\cup C$ *is convex.*

Proof. For (b) iff (c), cf. Eggleston [1], pp. 9–10.

The straight line $f(x, y)$ determined by the points x and y disconnects R^2 into two convex, open subsets $A(x, y)$ and $B(x, y)$. If $f(x, y)$ is a supporting line of C, we shall always assume that $C-f\subset A(x, y)$.

LEMMA 2. *C is convex iff $f(x_{i-1}, x_i)$ is a supporting line of C, $i=1, \cdots, n$.*

Proof. We may use Theorems 8 and 9 from Eggleston [1], pp. 20–21 to prove that $\mathrm{Cl}\, b(C)$ is convex iff $f(x_{i-1}, x_i)$ is a supporting line of C, $i=1, \cdots, n$. Lemma 2 then follows from Lemma 1.

Let x and y be distinct points of R^2 and $S\subseteq R^2$. x is said to have access to y rel S, or, equivalently, y is accessible rel S to x, if $(\overline{xy}-\{x, y\})\cap S=\varnothing$.

The vertices x_i and x_j of C are said to be adjacent if $\overline{x_ix_j}\subset C$, but no vertex x_k of C, $k\neq i$ or j, is contained in $\overline{x_ix_j}$. We set $x_{n+1}=x_1$ and $x_{-1}=x_{n-1}$.

LEMMA 3. *If $x\in R^2-C$, C is convex, and x has access to x_i rel C, then at least one vertex of C adjacent to x_i is accessible rel $C\cup\overline{xx_i}$ to x.*

Proof. $\overline{xx_i} \subset f(x, x_i)$; *therefore if* $y \notin f(x, x_i)$, *then* $\overline{yx} \cap \overline{xx_i} = \{x\}$. Hence for some vertex x_j of C adjacent to x_i to be accessible rel $C \cup \overline{xx_i}$ to x, it is necessary and sufficient that x_j be accessible rel C to x.

Case 1. $x \in b(C)$. All vertices of C are accessible rel C to x; hence Lemma 3 is trivial.

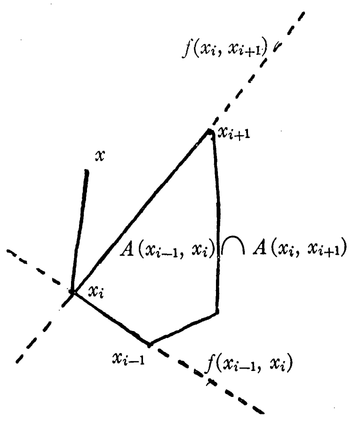


FIG. 1.

Case 2. $x \in u(C)$. By convention $C - (\overline{x_ix_{i+1}} \cup \overline{x_{i-1}x_i}) \subset A(x_i, x_{i+1}) \cap A(x_{i-1}, x_i)$. It is clear that $x \notin A(x_i, x_{i+1}) \cap A(x_{i-1}, x_i)$ since this would give a contradiction to x_i accessible rel C to x. If $x \in B(x_i, x_{i+1})$, then, by the convexity of $B(x_i, x_{i+1})$, x_{i+1} is accessible rel C to x. If $x \in B(x_{i-1}, x_i)$, then x_{i-1} is accessible rel C to x.

LEMMA 4. If $x \in R^2 - C$ and C is convex, then at least one vertex of C is accessible rel C to x.

Proof. Case 1. $x \in b(C)$. Trivial.

Case 2. $x \in u(C)$. Since $\{\bigcap_{i=1}^{n} A(x_{i-1}, x_i)\} \cup C = \mathrm{Cl}\, b(C)$ contains $b(C)$, x must be in $B(x_{i-1}, x_i)$ for some i; hence both x_{i-1} and x_i are accessible rel C to x.

LEMMA 5. *Let* $C_1 = [x_0, x_1, \cdots, x_{n-1}, x_n = x_0]$ *and* $C_2 = [y_0, \cdots, y_m = y_0]$ *be convex s.c.p. paths such that* $C_2 \subset b(C_1)$. *Then* (a) *for any vertex* y_j *of* C_2, *there is a vertex* x_i *that is accessible rel* C_2 *to* y_j; *and* (b) *if* x_i *is accessible rel* C_2 *to* y_j, *then some vertex of* C_1 *adjacent to* x_i *is accessible rel* $C_2 \cup \overline{x_iy_j}$ *to some vertex to* C_2 *adjacent to* y_j.

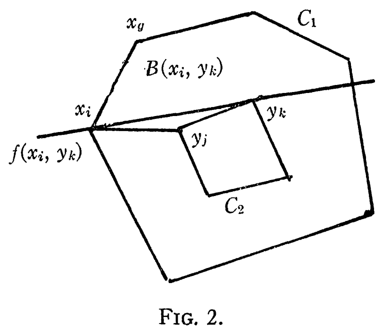


FIG. 2.

Proof. (a) Let f be any supporting line of C_2 through y_j. Then f divides $\mathrm{Cl}\, b(C_1)$ into convex parts A and B whose intersection is $f \cap \mathrm{Cl}\, b(C_1)$. We may suppose $\mathrm{Cl}\, b(C_2) \subseteq A$. Some vertex x_i of C_1 is in B; hence x_i is accessible rel C_2 to y_j.

(b) By Lemma 3 we may suppose x_i is accessible rel C_2 to both y_j and y_k where y_k is adjacent to y_j. $f(x_i, y_k)$ is a supporting line of C_2, and $\overline{y_j x_i}$ is contained in the same component of $R^2 - f(x_i, y_k)$ as $\mathrm{Cl}\, b(C_2) - f(x_i, y_k)$; hence we may find x_q adjacent to x_i accessible rel $C_2 \cup \overline{x_i y_j}$ to y_k.

Proof of proposition 1. We may suppose $S = S_1 \cup S_2 \cup \cdots \cup S_m$, with $S_i = \{x_{i1}, \cdots, x_{ip_i}\}$, $i = 1, \cdots, m$ such that:

(i) For $i = 1, \cdots, m-1$, $C_i = [x_{i1}, \cdots, x_{ip_i}, x_{i1}]$ is the s.c.p.p. which is the frontier of the convex hull of S_i;

(ii) $b(C_i)$ properly contains $\bigcup_{j>i} S_j$;

(iii) If the points of S_m are noncollinear, then the s.c.p.p. which forms the frontier of S_m is $C_m = [x_{m1}, \cdots, x_{mp_m}, x_{m1}]$;

(iv) If S_m consists of more than one point and the points of S_m are collinear, then the convex hull of S_m is $\overline{x_{m1} x_{mp_m}}$.

We must now produce a s.c.p.p. for which S is the set of vertices. If $S = S_1$, i.e., if $m = 1$, set $C = C_1$. Suppose $m \neq 1$.

Case 1. The points of S_2 are collinear; hence $m = 2$. If $S_2 = \{x_{21}\}$, set $C = (C_1 - \overline{x_{11}x_{12}}) \cup (\overline{x_{11}x_{21}} \cup \overline{x_{21}x_{12}})$. Suppose S_2 contains at least two points. Then x_{21} has access rel C_1 to some vertex, say x_{11}, of C_1. Then using the argument of Lemma 5 we can show that x_{2p_2} has access rel $C_1 \cup \overline{x_{11}x_{21}}$ to some vertex adjacent to x_{11}, say x_{12}. Set $C = (C_1 - \overline{x_{11}x_{12}}) \cup (\overline{x_{11}x_{21}} \cup \overline{x_{21}x_{2p_2}} \cup \overline{x_{2p_2}x_{12}})$.

Case 2. The points of S_2 are not all collinear. x_{21} is accessible rel C_2 to some x_{1j}. Using Lemma 5 we have some vertex of C_2 adjacent to x_{21} (assume it is x_{22}) accessible rel $C_2 \cup \overline{x_{1j}x_{21}}$ to a vertex of C_1 adjacent to x_{1j} (say $x_{1(j+1)}$). Then set $C' = (C_1 \cup C_2 \cup \overline{x_{1j}x_{21}} \cup \overline{x_{22}x_{1(j+1)}}) - (\overline{x_{1j}x_{1(j+1)}} \cup \overline{x_{21}x_{22}})$. If $m = 2$, set $C = C'$. If $m \neq 2$, we again have several cases.

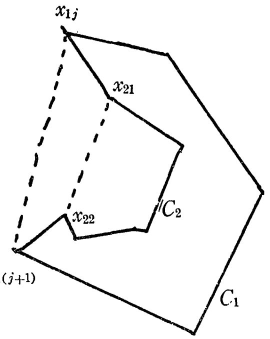


FIG. 3.

Case 1′. The points of S_3 are collinear; hence $m = 3$. If $S_3 = \{x_{31}\}$, set $C = (C' - \overline{x_{23}x_{22}}) \cup (\overline{x_{23}x_{31}} \cup \overline{x_{22}x_{31}})$. If S_3 contains more than one point, then it is easily seen that either x_{31} or x_{3p_3} (perhaps both) has access rel C_2 to some w_{2j}, $j \neq 1$, $j \neq 2$; suppose x_{31} has this property. Then x_{3p_3} has access rel $C_2 \cup \overline{x_{31}x_{2j}}$ to some vertex of C_2, say $x_{2(j+1)}$, adjacent to x_{2j}. Set $C=(C'-\overline{x_{2j}x_{2(j+1)}})\cup(\overline{x_{2j}x_{31}}\cup\overline{x_{31}x_{3p_3}}\cup\overline{x_{3p_3}x_{2(j+1)}})$.

Case 2'. The points of S_3 are noncollinear. x_{23} has access rel C_3 to some x_{3j}. We may again use Lemma 3 and suppose x_{24} has access rel $C_3\cup\overline{x_{3j}x_{23}}$ to $x_{3(j+1)}$. Set $C''=(C'\cup C_3\cup\overline{x_{23}x_{3j}}\cup\overline{x_{24}x_{3(j+1)}})-(\overline{x_{23}x_{24}}\cup\overline{x_{3j}x_{3(j+1)}})$. If $m=3$, set $C=C'$. If not, etc.

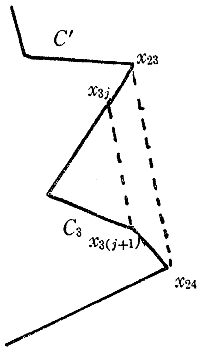


FIG. 4.

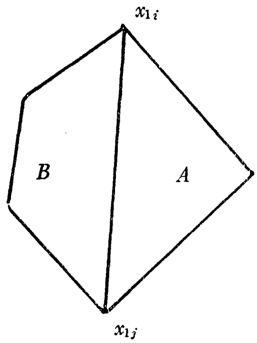


FIG. 5.

It should now be apparent that we can continue this procedure in all cases as far as need be to obtain a desired C. The process must terminate after m steps.

Proposition 2. If $m=1$ in proposition 1, then $p(S)=1$.

LEMMA 6. *Let* $C=[x_0, \cdots, x_{n-1}, x_n=x_0]$ *be convex. Then* $\overline{x_ix_j}\subset C$ *iff* $f(x_i, x_j)$ *is a supporting line of* C.

Proof. We may assume $i<j$. Then $x_{i+1}\in x_ix_j$; hence $f(x_i, x_{i+1})=f(x_i, x_j)$ is a supporting line of C by Lemma 2.

If $f(x_i, x_j)$ is a supporting line of C, then $\overline{x_ix_j}\subset f\cap C$ by the convexity of $b(C)\cup C$ (cf. Lemma 1).

Proof of proposition 2. We keep the notation of Proposition 1. If C^* is a s.c.p.p. for which $S_1=S$ is the set of vertices and $C^*\neq C_1$, then $C^*\subset \mathrm{Cl}\, b(C_1)$. Since $C^*\neq C_1$, we can find $\overline{x_{1i}x_{1j}}\subseteq C^*-C_1$. By Lemma 6, $f(x_{1i}, x_{1j})$ is not a supporting line of C_1; hence $\overline{x_{1i}x_{1j}}$ divides $\mathrm{Cl}\, b(C_1)$ into two convex parts A and B with $A\cap B=\overline{x_{1i}x_{1j}}$. It is clear that no point of A has access rel $\overline{x_{1i}x_{1j}}$ to any point of B. This immediately precludes $\overline{x_{1i}x_{1j}}$ from being in any s.c.p.p. for which S is the set of vertices. (See figure 5.)

The author closes by proposing a conjecture: Given a set all of whose points are not collinear, then of all the closed polygonal paths joining the points of S, the shortest is a s.c.p.p. with S as its set of vertices.

Reference

1. H. G. Eggleston, Convexity, Cambridge University Press, New York, 1958.

Now let $F_n = P_n + iQ_n$, which gives us

$$F_{n+1} = (1 - ix)F_n,$$
$$F_0 = b(x) + i[b(x) - a(x)]/x$$

and

$$F_n = (1 - ix)^n F_0,$$

where P equals the real part of F_n.

(Quickies on page 76)

ON COMPLEMENTING SETS OF NONNEGATIVE INTEGERS

A. M. VAIDYA, Bombay, India

1. Definition. *Two sets A and B of nonnegative integers are called complementing sets if*

(i) $A \cap B = \{0\}$, *and*

(ii) *every positive integer has a unique representation in the form $a+b$ for some $a \in A$, $b \in B$.*

2. Let m be any positive integer. Then the following pairs of sets are complementing.

Example 1. $A = \{0, 1, 2, 3, 4, \cdots\}$, $B = \{0\}$.

Example 2. $A = \{0, 1, 2, \cdots, m-1\}$, $B = \{0, m, 2m, 3m, \cdots\}$.

Example 3. $A = \{2km, 2km+1, \cdots, 2km+m-1 \mid k = 0, 1, 2, \cdots\}$, $B = \{0, m\}$.

Example 4. $A = \{0, 1, 2, \cdots, m-1, 2m, 2m+1, \cdots, 2m+m-1\}$, $B = \{0, m, 4m, 5m, 8m, 9m, \cdots, 4km, (4k+1)m, \cdots\}$.

In this note, we prove some of the properties of complementing sets suggested by these examples.

3. In the following, we assume that A and B are complementing sets. We exclude the trivial Example 1 of Section 2 from consideration. That is, we assume that both A and B contain positive integers. If an integer n is contained in A or B, then we say, "n appears." Obviously, 1 appears. We assume that $1 \in A$. We let m denote the least positive integer in B. Then $0, 1, 2, \cdots, m-1$ all appear and are in A; 0 and m appear and are in B. It is also clear that the integers $m+1, m+2, \cdots, 2m-1$ do not appear.

4. Lemma. *For an integer q and an integer r satisfying $1 \leqq r \leqq m-1$, if the integer $qm+r$ appears then so does qm and both are in A. Also if $qm \in A$, then $qm+r \in A$ for each r satisfying $1 \leqq r \leqq m-1$.*

Proof. We prove the lemma by induction on q. The results are clearly true for $q=0$. Suppose they are true up to $q-1$. Notice that this assumption implies that all integers in B which are less than qm are multiples of m.

Now suppose $qm+r$ appears (by our remark at the end of Section 3, $q>1$), but qm does not. Then we must have $qm=a+b$, where $0<a<qm$, $0<b<qm$, $a\in A$, $b\in B$.

As noted above, b is a multiple of m, so, for some integer t, $0<t<q$, we must have $a=tm$ and $b=(q-t)m$. Now $t\leqq q-1$ and $tm\in A$; hence by the induction hypothesis, $tm+r\in A$. But then $qm+r$ already has the representation

$$qm + r = (tm + r) + [(q - t)m]$$

in the form $a+b$, and so $qm+r$ cannot appear. Thus if $qm+r$ appears, then so does qm. Next, if $qm\in B$, then again $qm+r$ would have the representation

$$qm + r = (r) + qm$$

in the form $a+b$, and so $qm+r$ would not appear. Thus $qm\in A$.

Now suppose $qm\in A$ but $qm+r$ does not appear for some r, $1\leqq r\leqq m-1$. Then again, $qm+r=a+b$, where $0<a<qm+r$, $0<b<qm+r$, $a\in A$, $b\in B$.

Now b is a multiple of m or $b>qm$. If $b=qm+i$, $0<i<r$, then the integer $qm+m$ has the two representations

$$qm + m = (m - i) + (qm + i) = (qm) + (m)$$

in the form $a+b$, which is a contradiction. Therefore we have

$$a = tm + r, \qquad b = (q - t)m,$$

for some $t<q$.

Now $tm+r\in A$ and so, by the induction hypothesis, $tm\in A$. But then qm has the representation

$$qm = (tm) + [(q - t)m],$$

in the form $a+b$ and so qm would not appear. Thus if $qm\in A$, then $qm+r$ appears for each integer r satisfying $1\leqq r\leqq m-1$.

Finally, $qm+r\in A$, for, if $qm+r\in B$, then again $qm+m$ would have two representations

$$qm + m = (qm) + (m) = (m - r) + (qm + r),$$

in the form $a+b$. This completes the proof.

An immediate consequence of this lemma is the

THEOREM. *If A and B are complementing sets, then one of them contains only multiples of some fixed positive integer.*

Remark. The above theorem also holds in Example 1 of Section 2.

5. It would be noticed that in each of the four examples of complementing sets given in Section 2, one of the sets is finite. It seems difficult to determine if this is true in general.

VARIOUS PROOFS OF NEWTON'S THEOREM

KAIDY TAN, Fukien Normal College, Foochow, Fukien, China

This article is taken from my lecture on Geometrical Methods which I have been accustomed to give to the students in Fukien Normal College for the past several years. But it appears here with some additions and improvements. We apply both the pure geometrical methods and analytic methods. It contains more than twenty proofs; some are new, and the others are taken from eminent books and periodicals as shown in the bibliography. It has been proved by experience that different proofs of a theorem can instruct the students to apply the various theorems ingeniously, and to be skillful in solving problems, and also can stimulate the interest of investigation, and to forget the difficulties in it. Moreover, it is of great value in training the coming geometry teachers to master the various methods of geometry, either the advanced methods or elementary. That is the goal of this paper.

Now, we shall begin to discuss the proofs of the theorem.

THEOREM. *The mid-points of the three diagonals of a complete quadrilateral are collinear.*

For the sake of briefness in the statements, we adopt the following notations:

⊏⊐ quadrilateral $\qquad$ ▱ parallelogram.

$M \cup N$—the line passing through the points M and N.
$M \cup N \cup P$—the three points $M.N.P$ are collinear.
$AB \cap CD = P$—the lines AB, CD intersect at the point P.
$AB \cap CD \cap EF = P$—the three lines AB, CD, EF are concurrent at P.
$A \subset PQ$—the point A lies on the line PQ.
$PQ \supset A$—the line PQ passes through the point A.
∽—is similar to (this notation is derived from the first letter of Latin Similitudo)
∽̸—is homothetic to.

Let $ABCD$ be a ⊏⊐, and $AB \cap CD = E$, $BC \cap DA = F$. If L, M, N, are the mid-points of the three diagonals BD, AC, EF respectively, then $L \cup M \cup N$.

The line LMN is often called the Newton line of the ⊏⊐; some call it the diameter of the ⊏⊐. This theorem is also attributed to Gauss (see proof 16).

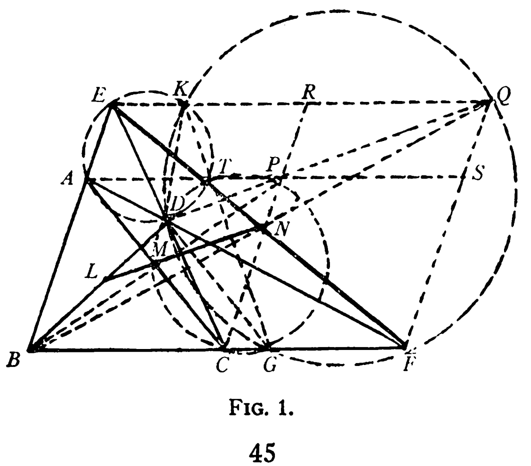


FIG. 1.

Proof 1. Construct the $\square EBFQ$; draw $AS \| BF$, $CR \| BE$, $S \subset FQ$, $R \subset EQ$; let $AS \cap CR = P$. Through D, F, Q draw the circle intersecting EQ, BF at K, G respectively. Join KD (Fig. 1). $\angle ADK + \angle AEK = \angle FQK + \angle AEK = 180°$ $\therefore$ E, A, D, K are concyclic. Let this circle intersect AS at T and join TD, DG. $\because$ $EK \| AT$, $\therefore$ $\widehat{EA} = \widehat{KT}$, $\angle EAT = \angle KTA$, $\therefore$ $\angle EDT = \angle EAT = \angle KTA = \angle KGB$, $\therefore$ D, C, G, T are concyclic, and $\angle APC = \angle EAT = \angle EDT$, $\therefore$ T, P, D, C are concyclic, $\therefore$ D, C, G, P, T are concyclic, $\angle TPD = \angle TGD = \angle KGD = \angle KQD$. $\because$ $EQ \| AP$, $\therefore$ DP, DQ cannot but coincide, viz. $D \cup P \cup Q$. Hence the mid-points of BD, BP, BQ, i.e., the mid-points of BD, AC, EF, are collinear.

Note. This proof is easy to understand; it applies only the simple properties of the circle.

Proof 2. Apply the area theorem.

Join EL, EM, LC, LA, LF, FM (Fig. 2). Thus we have

$$\text{(1)} \quad \begin{aligned} \triangle ELM &= \triangle LCE - (\triangle CLM + \triangle CME). \text{ But} \\ \triangle LCE &= \triangle LCD + \triangle LDE = \tfrac{1}{2}\triangle BCD + \tfrac{1}{2} \end{aligned}$$

$\triangle BDE = \frac{1}{2}BCE$, and $\triangle CLM = \frac{1}{2}\triangle CLA$, $\triangle CME = \frac{1}{2}\triangle CAE$. Substituting in (1), we obtain

$$\triangle ELM = \tfrac{1}{2}\triangle BCE - \tfrac{1}{2}(\triangle CLA + \triangle CAE) = \tfrac{1}{2}(\text{concave} \square BCLA) = \tfrac{1}{4}\square ABCD$$

Similarly, we can prove $\triangle FLM = \frac{1}{4}\square ABCD$. $\therefore$ $ELM = \triangle FLM$.

If we draw EP, FQ perpendicular to LM and P, Q are the feet of perpendiculars, then $EP = FQ$. If $LM \cap EF = N'$, then $rt\triangle EPN' \equiv rt\triangle FQN'$. $\therefore$ $EN' = FN'$. $\therefore$ $N' \equiv N$. $\therefore$ $L \cup M \cup N$.

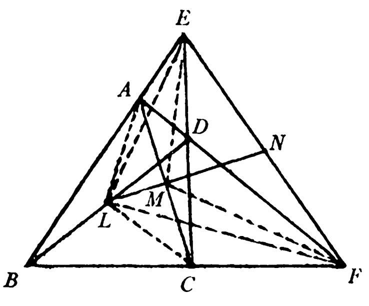


FIG. 2.

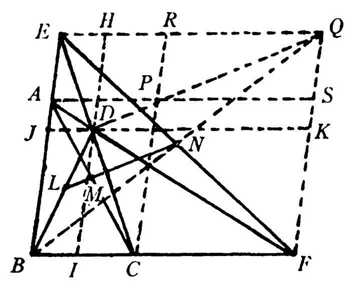


FIG. 3.

Proof 3. Alternate method of applying the area theorem.

Construct the $\square EBFQ$; draw $AS \| BF$, $CR \| BE$, $S \subset FQ$, $R \subset EQ$, and let $CR \cap AS = P$. Through D draw $HI \| EB$, $JK \| BF$, $H \subset EQ$, $I \subset BF$, $J \subset BE$, $K \subset FQ$ (Fig. 3). Then $\square DR = \square BD$, $\square DS = \square BD$ (by the theorem of the complement of a $\square$), $\therefore$ $\square DR = \square DS$. Subtracting $\square DP$ from both sides of this expression, we get $\square PH = \square PK$, $\therefore$ $D \cup P \cup Q$. Hence the mid-points of BD, BP, BQ are collinear, i.e., the mid-points of BD, AC, EF are collinear.

Proof 4. Apply the locus theorem concerning areas.

LEMMA 1. *A point P moves so that the sum or difference of the areas of the two triangles APB, CPD subtended at two fixed line-segments AB, CD remains constant; its locus is a straight line* [(4, Ex. 398)].

Join AL, AN, NC, CL, BM, BN, DM (Fig. 4). Then we have

$$\triangle LAB + \triangle LCD = \tfrac{1}{2}\triangle ABD + \tfrac{1}{2}\triangle BCD = \tfrac{1}{2}\square ABCD.$$

Similarly, $\triangle MAB + \triangle MCD = \frac{1}{2}\square ABCD$. Again, since N is the mid-point of EF, we have

$$\triangle NAB = \tfrac{1}{2}\triangle FAB,\ \triangle NCD = \triangle FCD,$$

$$\therefore\ \triangle NAB - \triangle NCD = \tfrac{1}{2}\square ABCD.$$

$$\therefore\ \triangle LAB + \triangle LCD = \triangle MAB + \triangle MCD = \triangle NAB - \triangle NCD,$$

$$\therefore\ L \cup M \cup N.$$

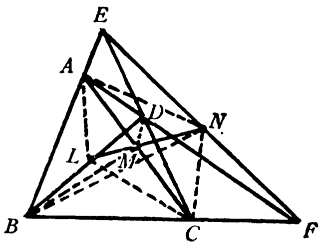


FIG. 4.

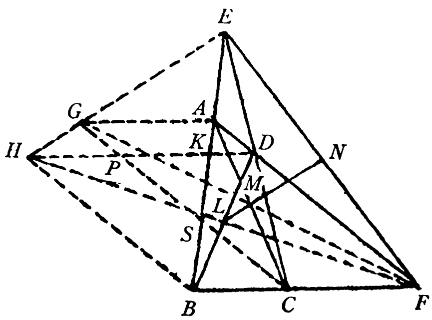


FIG. 5.

Proof 5. By similar figures.

Construct the $\square BFDH$ and $\square CFAG$; let $CG \cap BE = S$, $DH \cap BE = K$. (Fig. 5). Then we have

$$\frac{EA}{AS} = \frac{ED}{DC} = \frac{EK}{KB}, \qquad \therefore\ \frac{EA}{EK} = \frac{AS}{KB}. \tag{2}$$

Again, from $\triangle SAG \backsim \triangle BKH$, we get

$$\frac{AG}{KH} = \frac{AS}{KB} \tag{3}$$

From (2) and (3), we obtain

$$\frac{AG}{KH} = \frac{EA}{EK}.$$

Since $\angle HKE = \angle GAE$, we have $\triangle EGA \backsim \triangle EHK$, $\therefore\ E \cup G \cup H$. Hence the mid-points of FH, FG, FE are collinear, viz. the mid-points of BD, AC, EF are collinear.

Proof 6. By homothetic figures.

Construct the $\square EDFK$ and $\square CDAH$; then the diagonals DK, DH must intersect EF, AC at N, M respectively. Join AH, CH and produce them to meet BC, BA at P, Q respectively. Join QP, BH, HK (Fig. 6). Then we have

$$\frac{BP}{BC}=\frac{BA}{BE},\qquad \frac{BC}{BF}=\frac{BQ}{BA}.$$

Multiplying together, we get

$$\frac{BP}{BF}=\frac{BQ}{BE},\qquad \therefore\ PQ\|EF.$$

Again, $KF\|EC\|HP$, $EK\|AF\|QH$. $\therefore$ $\triangle KEF \not\sim \triangle HQP$; hence the lines joining corresponding vertices of these two triangles are concurrent, i.e.,

$$FP\cap KH\cap EQ=B.\qquad \therefore\ B\cup H\cup K.$$

Accordingly the mid-points of DB, DH, DK are collinear, viz. the mid-points of BD, AC, EF are collinear.

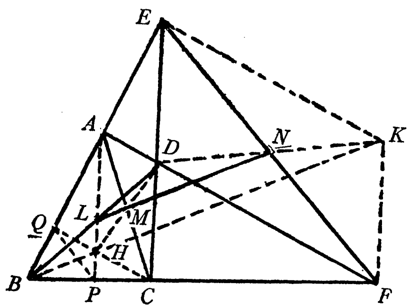


FIG. 6.

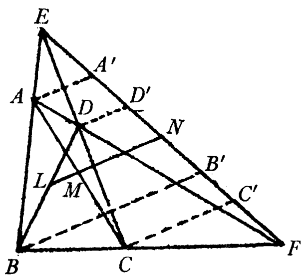


FIG. 7.

Proof 7. Apply parallel projection.

Join LM and produce it to meet EF at N'. From A, B, C, D draw the parallels to LM meeting EF at A', B', C', D' respectively (Fig. 7). Then we get

$$\frac{\triangle BCE}{\triangle BFA}=\frac{BC\cdot EB}{BF\cdot AB}=\frac{B'C'}{B'F}\cdot\frac{EB'}{A'B'};\qquad \frac{\triangle BFA}{\triangle CFD}=\frac{BF\cdot FA}{CF\cdot FD}=\frac{B'F}{C'F}\cdot\frac{FA'}{FD'};$$

$$\frac{\triangle CFD}{\triangle ADE}\quad\frac{FD\cdot DC}{AD\cdot DE}=\frac{FD'}{A'D'}\cdot\frac{D'C'}{D'E};\qquad \frac{\triangle ADE}{\triangle BCE}=\frac{DE\cdot EA}{CE\cdot EB}=\frac{D'E}{C'E}\cdot\frac{EA'}{EB'};$$

Multiplying together these four expressions, we have

$$1=\frac{B'C'\cdot FA'\cdot D'C'\cdot EA'}{A'B'\cdot C'F'\cdot A'D'\cdot C'E}. \tag{4}$$

But $N'C'=N'A'$, $N'B'=D'N'$, $\therefore$ $B'C'=A'D'$ $\therefore$ $D'C'=A'B'$. Substituting in (4) and simplifying, we have

$$1 = \frac{FA' \cdot EA'}{C'F \cdot C'E} \cdot$$

Hence $C'F \cdot C'E = FA' \cdot EA'$, so we have

$$(N'F - N'C')(N'E + N'C') = (N'F + N'A')(N'E - N'A'),$$

i.e., $-N'C'^2 + N'F \cdot N'C' - N'C' \cdot N'E = -N'F \cdot N'A' + N'A' \cdot N'E - N'A'^2$. But $N'C' = N'A'$; substituting in the above expression, we get

$$N'F - N'E = - N'F + N'E$$

$$\therefore \; 2N'F = 2N'E, \quad \text{and} \quad N'F = N'E,$$

i.e., N' is the mid-point of EF. $\therefore$ $N' \equiv N$, $\therefore$ $L \cup M \cup N$.

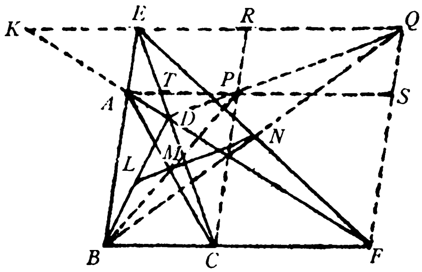


FIG. 8.

Proof 8. By the theorem on pencils.

Construct the $\square EBFQ$; draw $AS \| BF$, $CR \| BE$, $S \subset FQ$, $R \subset EQ$ and let $CR \cap AS = P$, $FA \cap QE = K$, $AS \cap EC = T$ (Fig. 8). Then

(5) $\quad AE \| FQ, \quad \therefore \; \dfrac{KE}{EQ} = \dfrac{KA}{AF} \cdot$

(6) $\quad EA \| PC, \quad \therefore \; \dfrac{AT}{TP} = \dfrac{ET}{TC} \cdot$

(7) $\quad KE \| AT \| CF, \quad \therefore \; \dfrac{KA}{AF} = \dfrac{ET}{TC} \cdot$

From (5), (6), (7) we obtain

$$\frac{KE}{EQ} = \frac{AT}{TP}, \qquad \therefore \; KA \cap ET \cap QP = D.$$

Hence $Q \cup P \cup D$; accordingly the mid-points of BQ, BP, BD are collinear, i.e., the mid-points of EF, AC, BD are collinear.

Proof 9. Apply Newton's locus theorem.

LEMMA 2. (*Newton's locus*) *On two fixed lines AX, BY there are two fixed points A, B and two variable points P, Q where $P \subset AX$, $Q \subset BY$, and $AP:BQ$ is a constant ratio $a:b$; a point N divides PQ in a constant ratio, $r:s$; then the locus of N is a straight line through M, where M divides AB in the constant ration $r:s$.*

This lemma has several proofs; we give only one as follows:

Construct the $\square ABQS$; then the locus of S is the line AZ. Join SP. Draw $NT \| QS$, $T \subset SP$ (Fig. 9). In the $\triangle ASP$, we have $AS:AP=$constant, $\therefore$ SP has a definite direction. Again, $PT:TS=PN:NQ=r:s$. Hence the locus of the division point T is a line AU. Now $AM:AB=PN:PQ$ (by hypothesis) $=NT:QS$. But $QS=AB$, $\therefore$ $NT=AM=$constant; therefore N lies on the line through M parallel to AU.

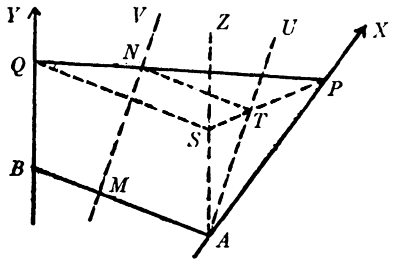


FIG. 9.

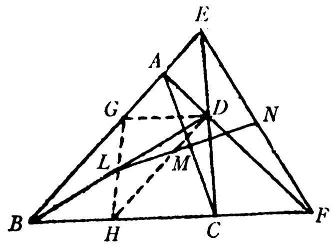


FIG. 10.

Now we can prove the original theorem as follows:

Construct the $\square DGBH$, where $G \subset AB$, $H \subset BC$. Let $GH \cap BD=L$ (Fig. 10). Now $\triangle AGD \backsim \triangle DHF$, $\therefore$ $GA:HD=GD:HF$; hence we have

$$GA \cdot HF = GD \cdot HD. \tag{8}$$

Similarly, $\triangle EGD \backsim \triangle DHC$; therefore we get

$$GE \cdot HC = GD \cdot HD. \tag{9}$$

From (8) and (9), we obtain $GA \cdot HF=GE \cdot HC$, or $GA:HC=GE:HF$. Now regard G, H as two fixed points on two fixed lines GE, HF, and $GA:HC$, $GE:HF$ as constant ratio; then the mid-points M, N of AC, EF lie on Newton's locus, i.e., $L \cup M \cup N$.

Proof 10. Apply Menelaus' Theorem.

Construct the $\square EBFQ$; draw $AS \| BF$, $CR \| BE$, $S \subset FQ$, $R \subset EQ$. Let $AS \cap CR=P$, $QP \cap AF=D'$ (Fig. 8). Now EC is a transversal of the $\triangle ABF$; hence we have

$$\frac{AD}{DF} \cdot \frac{FC}{CB} \cdot \frac{BE}{EA} = -1.$$

Again, QPD' is a transversal of the $\triangle SAF$; hence

$$\frac{AD'}{D'F} \cdot \frac{FQ}{QS} \cdot \frac{SP}{PA} = -1.$$

But $FC = SP$, $BE = FQ$, $CB = PA$, $EA = QS$,

$$\therefore \quad \frac{AD}{DF} = \frac{AD'}{D'F}, \qquad \therefore \quad D' \equiv D, \qquad \therefore \quad D \cup P \cup Q,$$

$\therefore$ the mid-points of BD, BP, BQ are collinear, i.e., the mid-points of BD, AC, EF are collinear.

Proof 11. Alternate proof.

Construct the $\square AFCG$ and $\square DFBH$; let $CG \cap DH = P$ (Fig. 5). Since BE is a transversal of the $\triangle CDF$,

$$\therefore \quad \frac{DA}{AF} \cdot \frac{FB}{BC} \cdot \frac{CE}{ED} = -1.$$

Now because $DAGP$, $PCBH$ are all $\square$s, from the properties of a $\square$, we obtain immediately that

$$\frac{PG}{GC} \cdot \frac{DH}{HP} \cdot \frac{CE}{ED} = -1;$$

hence from the $\triangle CDP$, we have $H \cup G \cup E$. $\therefore$ the mid-points of FH, FG, FE are collinear, i.e., the midpoints of BD, AC, EF are collinear.

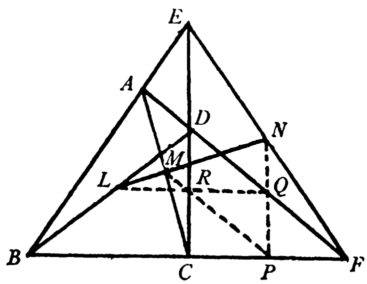


FIG. 11.

Proof 12. Alternate proof.

Let P, Q, R be the mid-points of CF, FD, DC; draw the $\triangle PQR$ (Fig. 11). $\because$ $QR \| FC$, $\therefore$ $QR \| FB$; hence $QR \supset L$. Similarly $PR \supset M$, $PQ \supset N$. Since $LQ \| BF$, $MP \| AF$, $PN \| CE$, we have

$$\frac{QL}{RL} = \frac{FB}{CB}, \qquad \frac{RM}{PM} = \frac{DA}{FA}, \qquad \frac{PN}{QN} = \frac{CE}{DE}.$$

$\because$ BE is a transversal of the $\triangle FCD$;

$$\therefore \quad \frac{FB}{BC} \cdot \frac{CE}{ED} \cdot \frac{DA}{AF} = -1.$$

Hence we have

$$\frac{QL}{LR}\cdot\frac{RM}{MP}\cdot\frac{PN}{NQ} = -1.$$

By the $\triangle PQR$ we obtain $L \cup M \cup N$.

Proof 13. Alternate proof.

Construct the $\square BECG$; draw $DH \| EB$, $AI \| EC$, $H \subset BG$, $I \subset CG$ (Fig. 12).

$$\therefore \frac{BH}{HG}\cdot\frac{GI}{IC}\cdot\frac{CF}{FB} = \frac{ED}{DC}\cdot\frac{BA}{AE}\cdot\frac{CF}{FB} = -1 \text{ } (\because AF \text{ is a transversal of the } \triangle EBC).$$

$\therefore$ H, I, F are collinear. Hence the mid-points of EH, EI, EF are collinear, i.e., the mid-points of BD, AC, EF are collinear ($\because$ $BEDH$, $AECI$ are all ▱).

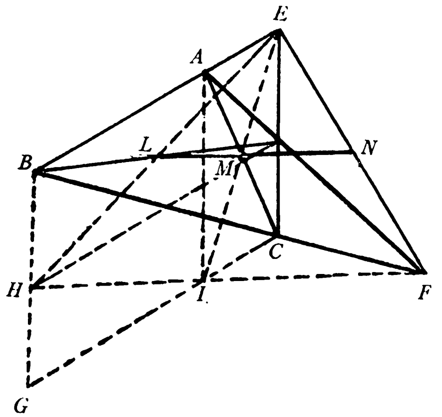


FIG. 12.

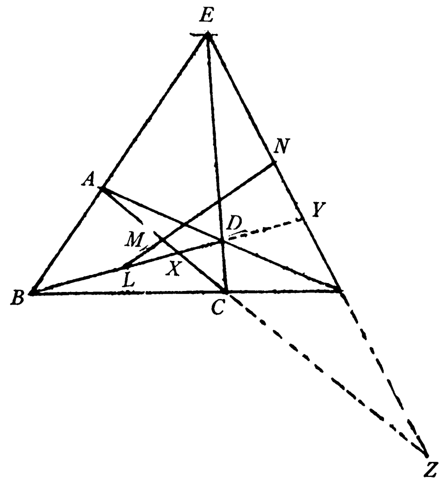


FIG. 13.

Proof 14. Apply the properties of a harmonic range. Let the three diagonals of the ▱ form a triangle XYZ where $X = AC \cap BD$, $Y = BD \cap EF$, $Z = AC \cap EF$ (Fig. 13). $\therefore$ $\{YZ, EF\} = -1$, since N is the mid-point of EF.

$$\therefore \frac{YN}{ZN} = \frac{EY^2}{EZ^2}.$$

Similarly,

$$\frac{ZM}{XM} = \frac{AZ^2}{AX^2}, \qquad \frac{XL}{YL} = \frac{BX^2}{BY^2}.$$

But BE is a transversal of the $\triangle XYZ$,

$$\therefore \frac{XB}{BY}\cdot\frac{YE}{EZ}\cdot\frac{ZA}{AX} = -1. \quad \therefore \frac{XL}{XL}\cdot\frac{YN}{ZN}\cdot\frac{ZM}{XM} = 1, \text{ or } \frac{XL}{LY}\cdot\frac{YN}{NZ}\cdot\frac{ZM}{MX} = -1,$$

$\therefore$ $L \cup M \cup N$.

Proof 15. Alternate proof.

As above,

$$\{BD, XY\} = -1, \qquad \therefore\ BL^2 = LX\cdot LY,$$

$$\therefore\ \frac{BX}{BY} = \frac{BL+LX}{BL+LY} = \frac{\sqrt{(LX\cdot LY)}+LX}{\sqrt{(LX\cdot LY)}+LY} = \frac{\sqrt{LX}(\sqrt{LY}+\sqrt{LX})}{\sqrt{LY}(\sqrt{LX}+\sqrt{LY})} = \frac{\sqrt{LX}}{\sqrt{LY}}\,.$$

Similarly,

$$\frac{AX}{AZ} = \frac{\sqrt{MX}}{\sqrt{MZ}}, \qquad \frac{EY}{EZ} = \frac{\sqrt{NY}}{\sqrt{NZ}}.$$

Since BE is a transversal of the $\triangle XYZ$,

$$\therefore\ \frac{XB}{BY}\cdot\frac{YE}{EZ}\cdot\frac{ZA}{AX} = -1, \text{ or } \frac{BX}{BY}\cdot\frac{EY}{EZ}\cdot\frac{AZ}{AX} = 1.$$

$$\therefore\ \sqrt{\left(\frac{LX}{LY}\cdot\frac{NY}{NZ}\cdot\frac{MZ}{MX}\right)} = 1, \text{ or } \frac{LX}{LY}\cdot\frac{NY}{NZ}\cdot\frac{MZ}{MX} = 1,$$

$$\therefore\ \frac{XL}{LY}\cdot\frac{YN}{NZ}\cdot\frac{ZM}{MX} = -1. \qquad \therefore\ L \cup M \cup N.$$

Proof 16. Apply the property of coaxial circles.

If the mid-points of the diagonals of the $\square$ are collinear, then the circles whose diameters are the diagonals of the $\square$ must be coaxial. Now, we may prove the latter theorem (Gauss' theorem).

Let BX, CY, EZ be the three altitudes of the $\triangle BCE$, and P_1 be the orthocenter of the triangle. Denote the three circles described on BD, AC, EF as diameters by $\odot L$, $\odot M$, $\odot N$, and the power of P_1 with respect to $\odot L$ by $[P_1]_L$. Now since $\angle BXD = \angle AYC = \angle EZF = 90°$. $\therefore \odot L \supset X, \odot M \supset Y, \odot N \supset Z$. Again, $[P_1]_L = P_1B\cdot P_1X$, $[P_1]_M = P_1Y\cdot P_1C$, $[P_1]_N = P_1Z\cdot P_1E$. But $P_1B\cdot P_1X = P_1Y\cdot P_1C$ ($\because$ B, X, C, Y are concyclic) $= P_1Z\cdot P_1E$($\because$ Y, C, E, Z are concyclic). $\therefore$ $[P_1]_L = [P_1]_M = [P_1]_N$, (Fig. 14).

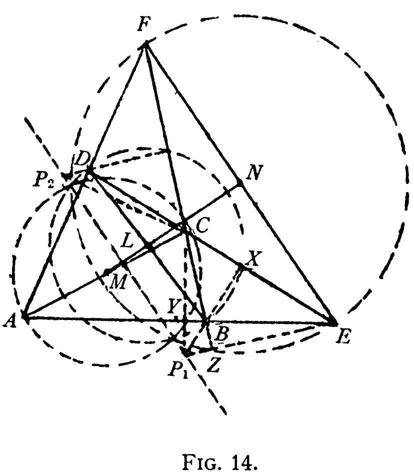


FIG. 14.

Similarly, if P_2 is the orthocenter of $\triangle CFD$, we can prove that

$$[P_2]_L = [P_2]_M = [P_2]_N.$$

$\therefore$ P_1P_2 is the common radical axis of the three circles $\odot L$, $\odot M$, $\odot N$; $\therefore$ the three circles are coaxial, and their centers must be collinear.

Proof 17. By the theory of involution.

Let the two circles described on AC, BD as diameters intersect at P, Q. Join P to A, B, C, D, E, F. Then $\angle APC = \angle BPD = 90°$. Since the lines joining any point with the vertices of a complete ⊏⊐ form a pencil in involution, this pencil $P\{AC, BD, EF\}$ will be in involution. Again, there are two pairs of conjugate lines at right angles; hence the other pair is at right angles too. $\therefore$ $\angle EPF = 90°$, $\therefore$ the circle described on EF as diameter must pass through P. Similarly, this circle passes through the second point Q. Therefore the three circles described on the diagonals as diameters are coaxial; it follows that their centers L, M, N are collinear (Fig. 15).

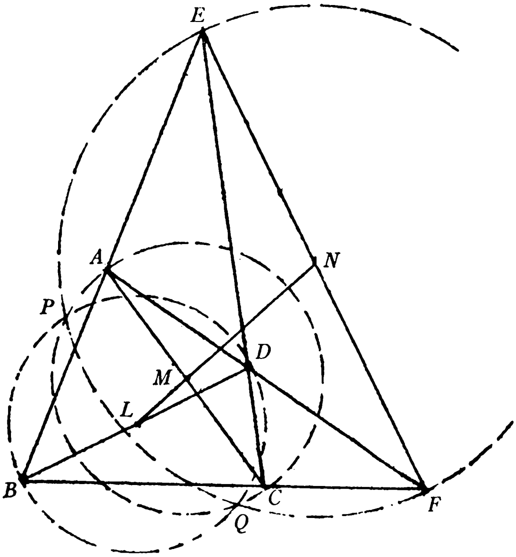


FIG. 15.

Proof 18. Coordinate method.

Take the two adjacent sides AB, AD for the axes, and let the coordinates of the vertices be $B(2a, 0)$, $E(2b, 0)$ $D(0, 2c)$, $F(0, 2d)$, as shown in Fig. 16. Then the equations of BF, DE are respectively

$$\frac{x}{a} + \frac{y}{d} = 2, \qquad \frac{x}{b} + \frac{y}{c} = 2.$$

Solving these simultaneous equations, we can get the coordinates of the point of intersection C. This can be derived from the following expressions:

$$\frac{x}{\dfrac{1}{c}-\dfrac{1}{d}}=\frac{y}{\dfrac{1}{a}-\dfrac{1}{b}}=\frac{2abcd}{bd-ac}.$$

$\therefore$ the coordinates of the mid-point M of AC are

$$\frac{(d-c)ab}{bd-ac},\quad \frac{(b-a)cd}{bd-ac}.$$

In proving $L \cup M \cup N$, we may prove that the following determinant vanishes:

$$\begin{vmatrix} a & c & 1 \\ b & d & 1 \\ (d-c)ab & (b-a)cd & bd-ac \end{vmatrix} \frac{1}{bd-ac}.$$

Multiply the first row by $-bd$, the second row by ac, and then add to the third row; then the elements of the new third row are all zero, and the determinant vanishes.

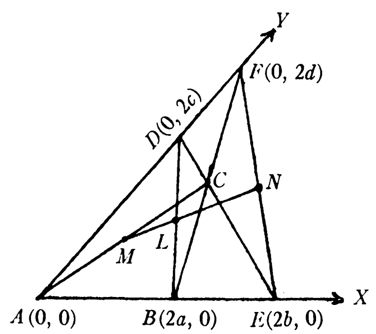


FIG. 16.

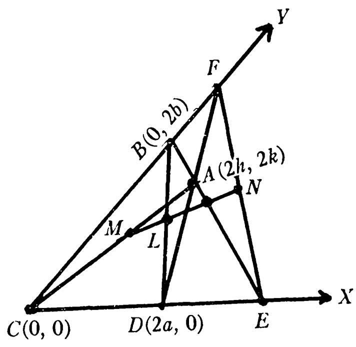


FIG. 17.

Proof 19. Alternate method.

Take the two adjacent sides BC, CD for the axes, and let the coordinates of the vertices be $D(2a, 0)$, $B(0, 2b)$, $A(2h, 2k)$ as shown in Fig. 17. Then the mid-points of BD, AC are $L(a, b)$, $M(h, k)$ respectively, and the equation of the line LM is

$$y-b=\frac{k-b}{h-a}(x-a)$$

or

(10) $$(h-a)y-(k-b)x=bh-ak$$

since the equation of AB is

(11) $$y-2b=\frac{k-b}{h}x.$$

Putting $y=0$ in (11), we have $x_E=-2bh/k-b$; $\therefore$ the coordinates of E are $(-2bh/k-b, 0)$. Similarly, the coordinates of F are $(0, -2ak/h-a)$. $\therefore$ the midpoint N of EF is $(-bh/k-b, -ak/h-a)$.

These coordinates clearly satisfy (10), i.e., $N\subset LM$.

N.B. There are some other coordinate methods we do not mention here. The reader may refer to [8].

Proof 20. Vector method.

See Fig. 16. Let $\mathbf{AB}=\mathbf{a}$, $\mathbf{AD}=\mathbf{b}$, $\mathbf{AE}=m\mathbf{a}$, $\mathbf{AF}=n\mathbf{b}$. Then $\mathbf{AL}=\mathbf{a}+\mathbf{b}/2$, $\mathbf{AN}=\frac{1}{2}(m\mathbf{a}+n\mathbf{b})$, $\mathbf{AM}=\frac{1}{2}\mathbf{AC}$. But $\mathbf{AC}=\mathbf{AB}+\mathbf{BC}=\mathbf{AB}+x\mathbf{BF}$; again, $\mathbf{AC}=\mathbf{AD}+\mathbf{DC}=\mathbf{AD}+y\mathbf{DE}$. Hence we have $\mathbf{a}+x(n\mathbf{b}-\mathbf{a})=\mathbf{b}+y(m\mathbf{a}-\mathbf{b})$ ($\because$ $\mathbf{BF}=\mathbf{AF}-\mathbf{AB}=n\mathbf{b}-\mathbf{a}$, $\mathbf{DE}=\mathbf{AE}-\mathbf{AD}=m\mathbf{a}-\mathbf{b}$).

Equating the coefficients of $\mathbf{a}$ and $\mathbf{b}$ respectively on both sides, we get

$$\begin{cases} 1-x=my, \\ nx=1-y. \end{cases}$$

Solving for x, we obtain

$$x=\frac{1-m}{1-mn}\cdot$$

$$\therefore\ \mathbf{AM}=\frac{1}{1}\mathbf{AC}=\frac{1}{2}\left\{\mathbf{a}+\frac{1-m}{1-mn}(n\mathbf{b}-\mathbf{a})\right\}=\frac{m(1-n)\mathbf{a}+n(1-m)\mathbf{b}}{2(1-mn)}$$

$$\therefore\ \mathbf{ML}=\mathbf{AL}-\mathbf{AM}=\frac{\mathbf{a}+\mathbf{b}}{2}=\frac{m(1-n)\mathbf{a}+n(1-m)\mathbf{b}}{2(1-mn)}=\frac{(1-m)\mathbf{a}+(1-n)\mathbf{b}}{2(1-mn)}.$$

$$\mathbf{MN}=\mathbf{AN}-\mathbf{AM}=\frac{m\mathbf{a}+n\mathbf{b}}{2}-\frac{m(1-n)\mathbf{a}+n(1-m)\mathbf{b}}{2(1-mn)}$$

$$=\frac{mn\{(1-m)\mathbf{a}+(1-n)\mathbf{b}\}}{2(1-mn)}=mn\mathbf{ML};\qquad \therefore\ M\cup L\cup N.$$

Proof 21. By the geometry of complex numbers.

Express the vertices of the complete quadrilateral $ABCDEF$ as the complex numbers a, b, c, d, e, f respectively (Fig. 16). Let m, n, k be real numbers. Then $c=b+m(f-b)=(1-m)b+mf$, $d=a+n(f-a)=(1-n)a+nf$. Again, $e=c+k(d-c)=(1-n)ka+(1-m)(1-k)b+\{m+(n-m)k\}f$. But $E\subset AB$,

$$\therefore\ m+(n-m)K=0,\ \text{i.e.,}\ k=\frac{m}{m-n}\cdot\qquad \therefore\ e=\frac{m(1-n)}{m-n}a+\frac{n(m-1)}{m-n}b.$$

Next, let the midpoints of AC, BD, EF be expressed by the complex numbers p, q, r; then

$$p=\frac{a+(1-m)b+mf}{2},\qquad q=\frac{b+(1-n)a+nf}{2},$$

$$r = \frac{1}{2}\left\{f + \frac{m(1-n)}{m-n}a + \frac{n(m-1)}{m-n}b\right\}.$$

But

$$\frac{p-q}{p-r} = \frac{\frac{1}{2}\{a + (1-m)b + mf - b - (1-n)a - nf\}}{\frac{1}{2}\left\{a + (1-m)b + mf - f - \frac{m(1-n)}{m-n}a - \frac{n(m-1)}{m-n}b\right\}}$$

$$= \frac{(m-n)\{na - mb + (m-n)f\}}{n(m-1)a + m(1-m)b + (m-1)(m-n)f} = \frac{m-n}{m-1} \text{ (a real number).}$$

$\therefore L \cup M \cup N.$

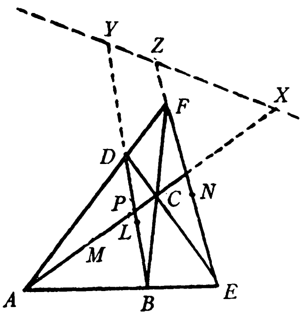


FIG. 18a.

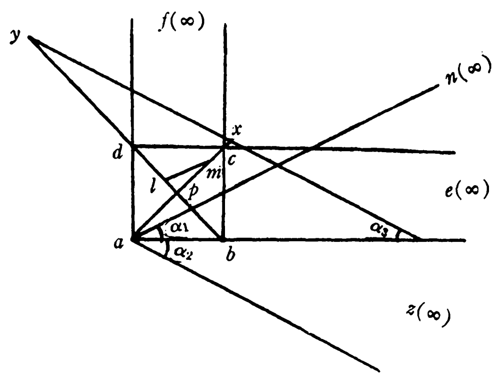


FIG. 18b.

Proof 22. By projective methods.

Let X, Y, Z be the intersections of AC, BD, EF with the line at infinity. Let $AC \cap BD = P$. Project EF to infinity and (at the same time) the angles FAC, EAC into angles of magnitude $\pi/4$. The result is shown in Fig. **18b**, in which small letters correspond to the capital letters of Fig. 18a.

$$(12) \qquad a\{enfz\} = A\{ENFZ\} = \{ENFZ\} = -1.$$

But $ae \perp af$, since $abcd$ is a square; hence $\alpha_1 = \alpha_2$ also $\alpha_2 = \alpha_3$ (alternate angles).

$$(13) \qquad \{amcx\} = \{bldy\} = -1;$$

and p is the midpoint of ac and bd.

Thus $pl \cdot py = pd^2 = pc^2 = pm \cdot px$, so that l, m, x, y are concyclic. Hence $\angle lmp = \angle lyx = \pi/4 - \alpha_3 = \pi/4 - \alpha_1 = \angle pan$.

Thus $ml \| an$, i.e., $l \cup m \cup n$; and hence $L \cup M \cup N$.

References

1. Akiyama, Plane Geometry (Lectures on Recent Higher Mathematics, VIII) Japanese ed., 1933, Tokyo.
2. C. V. Durell, Modern Geometry, St. Martin's Press, New York, 1949.
3. R. A. Johnson, Modern Geometry, Houghton Mifflin, New York, 1929, 1960.
4. Rouché & Comberousse, Traité de Géométrie, Ire partie, Géométrie Plane Nouv. éd. Gauthier-Villars, Paris, 1957.
5. R. Lachlan, Modern Pure Geometry, Macmillan, London, 1927.
6. L. S. Shively, Introduction to Modern Geometry, Wiley, New York, 1953.
7. E. H. Askwith, Pure Geometry, 2nd ed., Cambridge University Press, 1937.
8. S. L. Loney, Coordinate Geometry, parts I–II, Macmillan, London, 1945.
9. S. Kobayashi, Geometry of Complex Numbers, Japanese ed., Tokyo, 1954.
10. Mathematical Gazette, vol. XXXIX, No. 328, London (May 1955) p. 150.

THE ITERATION OF MEANS

LLOYD ROSENBERG, Hudson Laboratories of Columbia University and New York University

1. Introduction. With the increasing availability of electronic computers, iterative schemes have gained widespread acceptance. These techniques are very valuable but they do not always replace the need for mathematical analysis. If one can obtain an analytic expression for the quantity to which the iterative scheme converges, a great deal of computing time can be saved. Even more important, however, is the insight into the structure of the solution that can be gained from an analytic expression.

The iterative smoothing scheme presented in this paper was proposed by a very competent oceanographer as an "intuitively" reasonable smoothing procedure for his data. The initial calculations that were performed using this scheme were very slow in converging and excessive computation was required. The author was then asked to investigate methods for speeding up the convergence to avoid this expensive computation. It was possible for this smoothing procedure to avoid all iterative computations since an analytic expression for the smoothed value was obtained which was only a function of the original data. This result was very useful to the oceanographer since he could now determine how the procedure was smoothing the observations. In this case he found that his intuition had led him astray and the procedure was not doing what he had intended. This situation illustrates that although intuition is the starting point for many investigations, it should never be a substitute for logical thinking nor should large scale computations be thought of as anything more than an aid to well thought out procedures.

2. Statement of problem. Suppose that we are given a set of n numbers $x_1, x_2, \cdots, x_n$. From this set we form a new set $x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}$ where

$$x_1^{(1)} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad x_i^{(1)} = x_i, \qquad (i = 2, 3, \cdots, n).$$

References

1. Akiyama, Plane Geometry (Lectures on Recent Higher Mathematics, VIII) Japanese ed., 1933, Tokyo.
2. C. V. Durell, Modern Geometry, St. Martin's Press, New York, 1949.
3. R. A. Johnson, Modern Geometry, Houghton Mifflin, New York, 1929, 1960.
4. Rouché & Comberousse, Traité de Géométrie, Ire partie, Géométrie Plane Nouv. éd. Gauthier-Villars, Paris, 1957.
5. R. Lachlan, Modern Pure Geometry, Macmillan, London, 1927.
6. L. S. Shively, Introduction to Modern Geometry, Wiley, New York, 1953.
7. E. H. Askwith, Pure Geometry, 2nd ed., Cambridge University Press, 1937.
8. S. L. Loney, Coordinate Geometry, parts I–II, Macmillan, London, 1945.
9. S. Kobayashi, Geometry of Complex Numbers, Japanese ed., Tokyo, 1954.
10. Mathematical Gazette, vol. XXXIX, No. 328, London (May 1955) p. 150.

THE ITERATION OF MEANS

LLOYD ROSENBERG, Hudson Laboratories of Columbia University and New York University

1. Introduction. With the increasing availability of electronic computers, iterative schemes have gained widespread acceptance. These techniques are very valuable but they do not always replace the need for mathematical analysis. If one can obtain an analytic expression for the quantity to which the iterative scheme converges, a great deal of computing time can be saved. Even more important, however, is the insight into the structure of the solution that can be gained from an analytic expression.

The iterative smoothing scheme presented in this paper was proposed by a very competent oceanographer as an "intuitively" reasonable smoothing procedure for his data. The initial calculations that were performed using this scheme were very slow in converging and excessive computation was required. The author was then asked to investigate methods for speeding up the convergence to avoid this expensive computation. It was possible for this smoothing procedure to avoid all iterative computations since an analytic expression for the smoothed value was obtained which was only a function of the original data. This result was very useful to the oceanographer since he could now determine how the procedure was smoothing the observations. In this case he found that his intuition had led him astray and the procedure was not doing what he had intended. This situation illustrates that although intuition is the starting point for many investigations, it should never be a substitute for logical thinking nor should large scale computations be thought of as anything more than an aid to well thought out procedures.

2. Statement of problem. Suppose that we are given a set of n numbers $x_1, x_2, \cdots, x_n$. From this set we form a new set $x_1^{(1)}, x_2^{(1)}, \cdots, x_n^{(1)}$ where

$$x_1^{(1)} = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad x_i^{(1)} = x_i, \qquad (i = 2, 3, \cdots, n).$$

We define the kth set recursively by the relationships

$$x_j^{(k)} = \bar{x}^{(k-1)} = \frac{1}{n}\sum_{i=1}^{n} x_i^{(k-1)}, \qquad (j = k - n[k/n])$$

and

$$x_i^{(k)} = x_i^{(k-1)}, \qquad (i = 1, 2, \cdots, n,\ i \neq j)$$

where $[x]$ is the greatest integer not exceeding x.

The problem is to determine the existence of

$$\operatorname*{Lim}_{k\to\infty} x_i^{(k)}, \qquad (i = 1, 2, \cdots, n)$$

and, if it exists, to determine its value. It will be more convenient to reformulate the problem in terms of vectors and linear transformations. We define the vectors X and $X^{(k)}$ by $X=(x_1, x_2, \cdots, x_n)$, and $X^{(k)}=(x_1^{(k)}, x_2^{(k)}, \cdots, x_n^{(k)})$, $(k=1, 2, \cdots)$. Let T_i $(i=1, 2, \cdots, n)$ be an $n\times n$ matrix, whose rth row is denoted by $t_r^{(i)}$, where

$$t_i^{(i)} = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right),\quad t_r^{(i)} = (\delta_{1r}, \delta_{2r}, \cdots, \delta_{nr}),\quad (r = 1, 2, \cdots, n; r \neq i),$$

and δ_{ij} is the Kronecker delta. Thus

$$\begin{aligned} X^{(1)} &= T_1X, \\ X^{(2)} &= T_2X^{(1)} = T_2T_1X, \\ &\ \ \vdots \\ X^{(n)} &= T_nX^{(n-1)} = T_nT_{n-1}\cdots T_1X = TX, \end{aligned}$$

where $T=T_nT_{n-1}\cdots T_1$.

Now for any integer a

$$X^{(an)} = T^aX.$$

If we can show that the matrix T^a approaches a matrix L which has all rows equal, then $X^{(an)}$ will approach a vector with n equal elements. Since $X^{(an+j)} = T_jT_{j-1}\cdots T_1X^{(an)}$, $(j=1, 2, \cdots, n-1)$, it will also approach the same limit as $X^{(an)}$ because of the structure of the T_i's.

In the next section we will show that such a matrix $L=\{l_{ij}\}$ does exist and then determine its entries.

3. Existence and evaluation of the limit.

DEFINITION. *A matrix of nonnegative elements is said to be a stochastic matrix if the sum of the elements in each row is one.*

THEOREM 1. *The matrix*

$$L = \operatorname*{Lim}_{a\to\infty} T^a$$

exists and has all rows equal to $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$ *where* λ *is the unique solution to the system of equations* $\lambda=\lambda T$ *such that* $\sum_{i=1}^{n}\lambda_i=1$.

PROOF. Since the matrix $T=T_nT_{n-1}\cdots T_1$ is the product of stochastic matrices, it is a stochastic matrix.

If we let $S_k=T_kT_{k-1}\cdots T_1$ then $S_k=T_kS_{k-1}$, $(k=2, 3, \cdots, n)$ and $S_n=T$. From the structure of T_k it follows that S_k differs from S_{k-1} only in the kth row and hence it can be shown inductively that the elements in the first k rows of S_k are all positive. Since $S_n=T$ is a stochastic matrix with all positive elements, it corresponds to the transition probability matrix of an irreducible finite Markov chain and hence an ergodic chain. The conclusions of the theorem follow from a well-known theorem for ergodic Markov chains (cf. Feller [1] p. 356).

LEMMA 1. *The series*

$$\frac{j+1}{n}+\sum_{i=2}^{n-j}\frac{(j+i)(n+1)^{i-2}}{n^i}=1, \qquad (j=0, 1, 2, \cdots, n-2).$$

Proof. If we note that

$$\sum_{i=2}^{n-j}\frac{(j+i)(n+1)^{i-2}}{n^i}=\frac{1}{n(n+1)}\sum_{i=2}^{n-j}(j+i)\left(\frac{n+1}{n}\right)^{i-1}, \tag{1}$$

then by letting $x=(n+1)/n$, the right-hand side of (1) can be rewritten as

$$\sum_{i=2}^{n-j}(j+i)x^{i-1}=j\sum_{i=2}^{n-j}x^{i-1}+\sum_{i=2}^{n-j}ix^{i-1}. \tag{2}$$

Since the first sum on the right of (2) is the sum of a geometric series, and the second sum on the right of (2) is the sum of the derivative of a geometric series, the sum on the left-hand side of (2) is easily evaluated and the result of the lemma follows.

THEOREM 2. *The vector* $\lambda=(\lambda_1, \lambda_2, \cdots, \lambda_n)$, *where*

$$\lambda_s=\frac{2s}{n(n+1)}, \qquad (s=1, 2, \cdots, n)$$

is the unique solution to the system of equations $\lambda=\lambda\, T$ *such that* $\sum_{i=1}^n \lambda_i=1$.

Proof. The proof will be done inductively. First we will show that

$$\lambda_1=\sum_{i=1}^n \lambda_i t_{i,1}$$

where $t_{i,j}$ is the element in the ith row and jth column of T. Since the matrix T is the product of the matrices T_i, at each iteration only one row will be different from that of the previous iteration. One can easily show that

$$t_{1,1}=\frac{1}{n}, \qquad t_{2,1}=\frac{1}{n^2},$$

$$t_{k,1} = \frac{1}{n}(t_{1,1} + t_{2,1} + \cdots + t_{k-2,1}) + \frac{1}{n} t_{k-1,1} = t_{k-1,1} + \frac{1}{n} t_{k-1,1}, \qquad (k = 3, 4, \cdots, n).$$

Therefore it follows that

$$t_{k,1} = \frac{(n+1)}{n} t_{k-1,1} = \frac{(n+1)^{k-2}}{n^k}, \qquad (k = 3, 4, \cdots, n).$$

Thus we must show that

$$\frac{2}{n(n+1)} = \frac{2}{n(n+1)} \left\{ \frac{1}{n} + \sum_{i=2}^{n} i \frac{(n+1)^{i-2}}{n^i} \right\}.$$

But by Lemma 1 with $j=0$ we have that the expression inside the bracket is 1 and hence the equation holds.

Now let us assume that $\lambda_k = \sum_{i=1}^{n} \lambda_i t_{i,k}$ and show that $\lambda_{k+1} = \sum_{i=1}^{n} \lambda_i t_{i,k+1}$ for $1 \leqq k \leqq n-2$. From the method of forming the matrix T we have that

$$t_{i,k+1} = t_{i,k}, \qquad (i = 1, 2, \cdots, k),$$

$$t_{k+1,k+1} = t_{k+1,k} + \frac{1}{n_2},$$

and

$$t_{k+j,k+1} = t_{k+j,k} + \frac{(n+1)^{j-2}}{n^j}, \qquad (j = 2, 3, \cdots, n-k).$$

Now for $k \leqq n-2$ we have

$$\sum_{i=1}^{n} \lambda_i t_{i,k+1} = \sum_{i=1}^{n} \lambda_i t_{i,k} + \lambda_{k+1} \cdot \frac{1}{n} + \sum_{j=2}^{n-k} \lambda_{k+j} \frac{(n+1)^{j-2}}{n^j}$$

$$= \lambda_k + \frac{2}{n(n+1)} \left(\frac{k+1}{n} + \sum_{j=2}^{n-k} \frac{(k+j)(n+1)^{j-2}}{n^j} \right)$$

by applying the induction hypothesis. If we now apply Lemma 1 to the expression in the brackets we obtain

$$\sum_{i=1}^{n} \lambda_i t_{i,k+1} = \frac{2k}{n(n+1)} + \frac{2}{n(n+1)} = \frac{2(k+1)}{n(n+1)} = \lambda_{k+1}.$$

For $k = n-1$, we have similarly

$$\sum_{i=1}^{n} \lambda_i t_{i,n} = \sum_{i=1}^{n} \lambda_i t_{i,n-1} + \lambda_n \frac{1}{n}$$

$$= \frac{2}{n(n+1)} [n - 1 + 1] = \lambda_n.$$

Thus the theorem is proved.

If we have a sequence of numbers $x_1, x_2, \cdots, x_n$ and we perform the iterative procedure given above we obtain

$$\lim_{k\to\infty} x_i^{(k)} = \frac{2}{n(n+1)} \sum_{j=1}^{n} jx_j, \qquad (i = 1, 2, \cdots, n)$$

which is the solution to the given problem. The structure of the solution is now clearly a weighted average of the numbers where the weights are proportional to the integers. It also becomes apparent that the order in which x_i's are taken is very important. For the particular data being studied this was the great disadvantage of this smoothing procedure.

In general, an iterative smoothing scheme defines recursively a sequence of scalars, vectors, or functions. Therefore the problem of determining the limit of the procedure can be obtained by the usual methods of analysis. However, an additional complication is introduced by defining the general term of the sequence recursively rather than by an analytic expression. This additional complication makes the evaluation of the limit much more difficult, whereas the difficulty of determining the existence of a limit may only be slightly increased. In the scheme considered in this paper it was possible to determine a nonrecursive expression for the general term of the sequence and thus reduce the problem to a standard problem of analysis. In Theorem 1 we were able to determine the existence of a limit and a method of evaluating the limit in terms of the matrix T. However, for arbitrary n the matrix T was very complicated and a simple representation of the elements of the matrix was not possible. Thus it did not appear promising to try to evaluate these elements and solve directly the system of equations given in Theorem 1. Instead the system of equations was solved for small values of n and then the result of Theorem 2 was conjectured and proved inductively without ever explicitly evaluating the elements of the matrix T for arbitrary n. Since the purpose of this investigation was to determine the structure of the iterative scheme, an inductive proof of the result was quite satisfactory.

Acknowledgments. I wish to thank the reviewer for his many helpful suggestions in preparing the final version of this paper.

This work was partially supported by the Office of Naval Research under Contract Nonr-266(84). Reproduction in whole or in part is permitted for any purpose of the United States government. It is Hudson Laboratories of Columbia University Contribution No. 167.

Reference

1. W. Feller, An introduction to probability theory and its applications, 2nd ed., Wiley, New York, 1957.

ON THE SOLUTION OF THE QUARTIC

ROBERT W. PACKARD, University of Maine

Consider

$$x^4 + bx^3 + cx^2 + dx + e = 0, \tag{1}$$

where b, c, d, and e are real. It is well known that (1) can be written as

$$\left(x^2 + \frac{b}{2}x + \frac{p}{2}\right)^2 = \left(p + \frac{b^2}{4} - c\right)x^2 + \left(p\frac{b}{2} - d\right)x + \left(\frac{p^2}{4} - e\right), \tag{2}$$

where p is any complex number, and the right hand side of (2) can be written as

$$(qx + r)^2 \tag{3}$$

where q and r are complex numbers if and only if

$$\left(p\frac{b}{2} - d\right)^2 = 4\left(p + \frac{b^2}{4} - c\right)\left(\frac{p^2}{4} - e\right), \tag{4}$$

i.e., if and only if

$$F(p) \equiv p^3 - cp^2 + (bd - 4e)p + (4ce - b^2e - d^2) = 0. \tag{5}$$

The following proof shows that *real* p, q, and r can always be found. Note that

$$F\left(c - \frac{b^2}{4}\right) = -\left(d - \frac{b}{2}c + \frac{b^3}{8}\right)^2 \leqq 0. \tag{6}$$

Case I. If $F(c-b^2/4)<0$, then, since $F(p)$ becomes positive for p large enough, there exists a real $p>c-b^2/4$ such that (5) is satisfied. Under these circumstances $(p + b^2/4 - c) > 0$ and, from (4), $(p^2/4 - e) \geqq 0$. Thus $q=\sqrt{\{p+b^2/4-c\}}$ and $r=\pm\sqrt{\{p^2/4-e\}}$ are real.

Case II. If $F(c-b^2/4)=0$ and $p^2/4-e\geqq 0$ when $p=c-b^2/4$, then $p=c-b^2/4$ satisfies (5). Also, $p+b^2/4-c=0$ and, from (4), $p(b/2)-d=0$. Then $q=0$ and $r=\sqrt{\{p^2/4-e\}}$ are real.

Case III. If $F(c-b^2/4)=0$ and $p^2/4-e<0$ when $p=c-b^2/4$, then, from (6),

$$d - \frac{b}{2}c + \frac{b^3}{8} = 0 \tag{7}$$

and, substituting $p=c-b^2/4$ into $p^2/4-e<0$, we obtain

$$c^2 - \frac{b^2}{2}c + \frac{b^4}{16} - 4e < 0. \tag{8}$$

Thus

$$F'\left(c - \frac{b^2}{4}\right) = b\left(d - \frac{b}{2}c + \frac{b^3}{8}\right) + \left(c^2 - \frac{b^2}{2}c + \frac{b^4}{16} - 4e\right) < 0.$$

Therefore, (5) must be satisfied by a real value of $p>c-b^2/4$ and we have the same result as in Case I.

A PROOF OF THE FORMULA REPRESENTING THE LOGARITHM AS THE LIMIT OF A SEQUENCE

R. F. MATLAK, The University of New South Wales

One of the most interesting results of the theory of the logarithmic function is presented by the formula enunciated by E. Halley (1656–1742) according to which, for all real positive values of x,

$$\log x = \lim_{n \to \infty} n(\sqrt[n]{x} - 1).$$

However, as can be seen from [1; Ch. iv, Ex. xxvii, No. 10, p. 141, and Article 75, pp. 144–145; Ch. ix, Articles 215 and 216, pp. 410 and 411, resp.] and [2; Ch. vi, Section 21, 77, p. 252] this is rather difficult to establish in a rigorous fashion.

The following is a simple derivation of this result based on the definition of $\log x$ as the definite integral $\int_1^x (1/t)dt$.

First assume that $x > 1$. Given any integer n greater than unity, consider the sub-division of the interval $(1, x)$ defined by the finite monotonic increasing sequence $1, \sqrt[n]{x}, \sqrt[n]{x^2}, \cdots, \sqrt[n]{x^{n-1}}, x$.

Corresponding to this subdivision, let $S_n(x)$, $s_n(x)$ be the approximations to the value of the integral $\int_1^x (1/t)dt$ formed, respectively, for the maximum, minimum values of the function $1/t$ over the (closed) subintervals of the interval $(1, x)$. Evidently

$$S_n(x) = \sum_{k=1}^{n} \frac{1}{\sqrt[n]{x^{k-1}}} \cdot \sqrt[n]{x^{k-1}}(\sqrt[n]{x} - 1) = n(\sqrt[n]{x} - 1),$$

and

$$s_n(x) = \sum_{k=1}^{n} \frac{1}{\sqrt[n]{x^{k}}} \cdot \sqrt[n]{x^{k-1}}(\sqrt[n]{x} - 1) = n\left(1 - \frac{1}{\sqrt[n]{x}}\right).$$

Now the maximum length of an interval defined by the sub-division is the length of the last subinterval which is equal to $x(1 - 1/\sqrt[n]{x})$. By virtue of the fact that $\lim_{n\to\infty} \sqrt[n]{x} = 1$, this tends to zero as $n \to \infty$.

By the elementary theory of integration it therefore follows that

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} s_n(x) = \int_1^x \frac{1}{t}\,dt;$$

i.e., $\lim_{n\to\infty} n(\sqrt[n]{x} - 1) = \lim_{n\to\infty} n(1 - 1/\sqrt[n]{x}) = \log x$.

Now let $0 < x < 1$. Since $1/x$ is then greater than unity, and $\log 1/x = -\log x$, the previous argument evidently holds.

Finally, noting that the result is obviously valid when $x = 1$ concludes the proof.

Remark. It can be observed that, with $S_n(x)/s_n(x) = \sqrt[n]{x} \to 1$ as $n \to \infty$, the accuracy of the approximation to the value of $\log x$ attained by using the Halley formula can be determined without difficulty.

References

1. G. H. Hardy, A course of pure mathematics, 9th ed., Cambridge University Press, New York, 1948.
2. A. Ostrowski, Vorlesungen ueber Differential- und Integralrechnung, vol. 1, Verlag Birkhaeuser, Basel, 1945.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Albert Einstein and the Cosmic World Order. By Cornelius Lanczos. Interscience (Wiley), New York, 1965. vi+139 pp. $3.95.

In 1962 Professor Lanczos delivered a series of six lectures at the University of Michigan on "The Place of Albert Einstein in the History of Physics." This book resulted from those lectures.

The first chapter, the last chapter, and various references throughout the text eulogize Einstein. Einstein's ability to go to the core of problems and find a fundamental or unifying principle is repeatedly indicated. The emphasis on Einstein's place at the pinnacle of physics in the first part of the twentieth century sometimes leaves the impression that his discoveries resulted from adventures of the mind with very little outside stimulation. Indeed, the author may, in part, wish this impression to be made. However, he puts matters in proper perspective with excellent historical background.

Most of the book is concentrated upon the special and general theories of relativity. The postulates are stated and both physical and philosophic aspects of the theories are set forth. The mathematical background needed for each theory is indicated. The philosophic importance of Minkowski's interpretation of the special theory of relativity as a geometric theory is emphasized, as is the dependence of the general theory on tensor calculus. The description of the mathematical background of general relativity primarily is concerned with Gauss's contributions to the geometry of surfaces and Riemann's generalizations of the work. There is a trend of thought throughout the book which illustrates the role that mathematics plays in theoretical physics. Some of this thought is found in the interesting question-answer monologues at the end of each chapter.

Einstein's studies of Brownian motion, light emission, and special and general relativity are described in as little technical language as can be used and still retain meaning. The illustrations used in describing these phenomena are very good. For those who have no technical background, any meaningful description of fundamental ideas in physical science will be difficult to digest; however, such individuals should profit from reading this book. Students of mathematics, physics, and philosophy will find Professor Lanczos' book interesting and profitable reading.

R. C. WREDE, San Jose State College

References

1. G. H. Hardy, A course of pure mathematics, 9th ed., Cambridge University Press, New York, 1948.
2. A. Ostrowski, Vorlesungen ueber Differential- und Integralrechnung, vol. 1, Verlag Birkhaeuser, Basel, 1945.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Albert Einstein and the Cosmic World Order. By Cornelius Lanczos. Interscience (Wiley), New York, 1965. vi+139 pp. $3.95.

In 1962 Professor Lanczos delivered a series of six lectures at the University of Michigan on "The Place of Albert Einstein in the History of Physics." This book resulted from those lectures.

The first chapter, the last chapter, and various references throughout the text eulogize Einstein. Einstein's ability to go to the core of problems and find a fundamental or unifying principle is repeatedly indicated. The emphasis on Einstein's place at the pinnacle of physics in the first part of the twentieth century sometimes leaves the impression that his discoveries resulted from adventures of the mind with very little outside stimulation. Indeed, the author may, in part, wish this impression to be made. However, he puts matters in proper perspective with excellent historical background.

Most of the book is concentrated upon the special and general theories of relativity. The postulates are stated and both physical and philosophic aspects of the theories are set forth. The mathematical background needed for each theory is indicated. The philosophic importance of Minkowski's interpretation of the special theory of relativity as a geometric theory is emphasized, as is the dependence of the general theory on tensor calculus. The description of the mathematical background of general relativity primarily is concerned with Gauss's contributions to the geometry of surfaces and Riemann's generalizations of the work. There is a trend of thought throughout the book which illustrates the role that mathematics plays in theoretical physics. Some of this thought is found in the interesting question-answer monologues at the end of each chapter.

Einstein's studies of Brownian motion, light emission, and special and general relativity are described in as little technical language as can be used and still retain meaning. The illustrations used in describing these phenomena are very good. For those who have no technical background, any meaningful description of fundamental ideas in physical science will be difficult to digest; however, such individuals should profit from reading this book. Students of mathematics, physics, and philosophy will find Professor Lanczos' book interesting and profitable reading.

R. C. WREDE, San Jose State College

Shortest Paths, Variational Problems. By L. A. Lyusternik. Pergamon Press, New York, 1964. x+102 pp. $2.75.

This book introduces problems of maxima and minima in an elementary fashion. Its purpose is to further the reader's knowledge of concepts rigorously attacked in the calculus of variations. The book consists of two parts.

The first part relies primarily on intuition and elementary geometric information in developing facts about geodesics on surfaces. The first problem attacked is that of geodesics, or shortest paths, on developable surfaces. These surfaces, intuitively, can be cut and rolled out on a plane (a circular cylinder is such a surface). Therefore, the shortest paths on the surfaces correspond to lines in the plane. It has been my experience that students with a substantial number of mathematics courses in their background have little feeling for such matters. Therefore, the book makes a distinct contribution in this respect. Lyusternik also investigates geodesics on the sphere and surfaces of revolution and introduces concepts from classical differential geometry needed for more general study of geodesics. The ideas of differential geometry are of interest but the beginner probably will find them hard to assimilate.

The second half of the book, in part, continues the study of shortest paths, but developments are based on physical ideas rather than geometric ones. Relations between work, potential energy and tensions are coupled with Dirichlet's principle, i.e., "for a mechanical system the position of its minimum potential energy is a position of equilibrium," to attack problems of the equilibrium of systems of strings. A combination of physical and geometric facts is used in the discussion of problems such as that of determining the shortest path on a surface enclosing a given measure of area. The last chapter starts with the assumption of Fermat's principle, i.e., "in an optical medium, the path of light from a point A to a point B has the least optical length of all paths joining A and B." A variety of facts concerning curves is obtained from this principle.

The difficulty of the second part of the book will depend on one's background in physics. If certain concepts can be accepted intuitively, then the ideas readily follow. In any case the book is a definite contribution to the literature. The intuitive information set forth cannot help but raise the level of mathematical maturity of readers who have not previously given significant attention to problems of the calculus of variations.

R. C. WREDE, San Jose State College

God and Golem, Inc. By Norbert Wiener. The M.I.T. Press, Cambridge, 1964. ix+99 pp. $3.95.

The Golem, in Ancient Hebrew mythology, was Adam in the shapeless stage of his creation, before he became Man. In Jewish legends the Golem was the artificial mindless man created by the Cabalist Rabbi Löw of Prague at the end of the sixteenth-century.

"The machine," Norbert Wiener emphasises in this literate, quotable, and earnest book, "is the modern counterpart of the Golem."

Shortest Paths, Variational Problems. By L. A. Lyusternik. Pergamon Press, New York, 1964. x+102 pp. $2.75.

This book introduces problems of maxima and minima in an elementary fashion. Its purpose is to further the reader's knowledge of concepts rigorously attacked in the calculus of variations. The book consists of two parts.

The first part relies primarily on intuition and elementary geometric information in developing facts about geodesics on surfaces. The first problem attacked is that of geodesics, or shortest paths, on developable surfaces. These surfaces, intuitively, can be cut and rolled out on a plane (a circular cylinder is such a surface). Therefore, the shortest paths on the surfaces correspond to lines in the plane. It has been my experience that students with a substantial number of mathematics courses in their background have little feeling for such matters. Therefore, the book makes a distinct contribution in this respect. Lyusternik also investigates geodesics on the sphere and surfaces of revolution and introduces concepts from classical differential geometry needed for more general study of geodesics. The ideas of differential geometry are of interest but the beginner probably will find them hard to assimilate.

The second half of the book, in part, continues the study of shortest paths, but developments are based on physical ideas rather than geometric ones. Relations between work, potential energy and tensions are coupled with Dirichlet's principle, i.e., "for a mechanical system the position of its minimum potential energy is a position of equilibrium," to attack problems of the equilibrium of systems of strings. A combination of physical and geometric facts is used in the discussion of problems such as that of determining the shortest path on a surface enclosing a given measure of area. The last chapter starts with the assumption of Fermat's principle, i.e., "in an optical medium, the path of light from a point A to a point B has the least optical length of all paths joining A and B." A variety of facts concerning curves is obtained from this principle.

The difficulty of the second part of the book will depend on one's background in physics. If certain concepts can be accepted intuitively, then the ideas readily follow. In any case the book is a definite contribution to the literature. The intuitive information set forth cannot help but raise the level of mathematical maturity of readers who have not previously given significant attention to problems of the calculus of variations.

R. C. Wrede, San Jose State College

God and Golem, Inc. By Norbert Wiener. The M.I.T. Press, Cambridge, 1964. ix+99 pp. $3.95.

The Golem, in Ancient Hebrew mythology, was Adam in the shapeless stage of his creation, before he became Man. In Jewish legends the Golem was the artificial mindless man created by the Cabalist Rabbi Löw of Prague at the end of the sixteenth-century.

"The machine," Norbert Wiener emphasises in this literate, quotable, and earnest book, "is the modern counterpart of the Golem."

Wiener is for both Man and his machines, and very much against the "priests of power" and "gadget worshipers" who should be stoned out of the Temples of Learning. "There is a sin, which consists of using the magic of modern automatisation to further personal profit or let loose the apocalyptic terrors of nuclear warfare. If this sin is to have a name, let the name be Simony or Sorcery . . . " and he, too, cautions: " . . . remember, that in the game of atomic warfare, there are no experts . . . "

During the last years of his life Wiener attempted to evolve a philosophy of cybernetics, of mechanical information theory for nonmachine Man possessing a mind capable of working with "material that any computer would have to reject as formless." While such a philosophy still needs to be evolved, Wiener did leave a testament for Man and plea for human understanding (not merely quantification and explanation) of his nonhuman inventions. "Render unto Man the things which are Man's and unto the computer the things which are the computer's" is surely sound advice.

"Rendering-unto" and "withholding-from" are truly complex evaluations in a society of proliferating gadgetry. The individual needs to search for those artesian wells of internal clarity; frequently they are difficult even to recognise.

Wiener has contempt for the social scientists and economists who "have developed the habit of dressing up their rather imprecise ideas in the language of the infinitesimal calculus" because they are "jealous" of the power of mathematical physics "without quite understanding the intellectual attitudes that had contributed to this power. . . ." He compares these practitioners to "primitive peoples (who) adopt the Western modes of denationalised clothing and of parliamentarism out of a vague feeling that these magic rites and vestments will at once put them abreast of modern culture and technique. . . ."

He has much to say about Man and his machines, and the purpose of this review is to recommend that you spend an hour or two with this little book, perhaps bearing in mind a word of caution.

The desire to anthropomorphise, to lend human qualities to things not human, is one of the pitfalls of science-technology. Wiener is also caught in this pitfall when he writes of a "criterion of success" for the machine, especially in terms of game-theory. The von Neumann theory of pitting opponents in machine-gamesmanship seems to be basically askew, essentially for this reason: despite whatever complexity is designed into the machine, there is present a *singularity* of machine-function as opposed to the multiplicity of human function. The machine, whatever job it performs, does so in accordance with design. It is not subjected to the ten thousand nondesign miscellaneous minutiae and distractions of life. The machine, in short, does not have extra-machine worries. But what human worries can actually be termed extra-human?

Design-theory and design-purpose certainly apply to the machine, as does "a criterion of success." However, "a criterion of success" does not apply to Man. How can a criterion exist without a design-theory? And a design-theory for Man does not, as yet, exist, for Man. "Success" is essentially a quantifiable term, a term of explanation but not of understanding. The difference between explanation and understanding is enormous. As I wrote in a recent essay, "Man, Machines and Morality" (*Man on Earth*—vol. I/No. 3), the machine has now become a problem more of morality than of technology.

Wiener's "priests of power" and "gadget worshipers" are indeed the quantifiers of "success" and, by direct extension, they are the corrupters of the young.

Socrates took the hemlock for a good deal less.

S P R CHARTER, Olema, California

A Collection of Problems on a Course of Mathematical Analysis. By G. N. Berman. Translated from Russian by D. E. Brown, and edited by I. N. Sneddon. Pergamon Press, New York, 1965. ix+588 pp. $12.50.

This is an English translation of the tenth edition of a standard problem book first published in 1947. According to the book's own statement it is intended for students studying mathematical analysis within the framework of a Technical College course, and it is especially suited for use with the text, *A Course of Mathematical Analysis*, by A. F. Bermant. (For a review of the two-volume English translation of this work see this MAGAZINE 38 (1965) 175–176.) Professor Bermant edited the earlier editions of the *Collection of Problems*, and participated in the work of the revisions, the tenth edition excepted.

There are sixteen chapters. The first fifteen of these follow exactly the subject matter as arranged in the fifteen chapters of the text by Professor Bermant. The sixteenth chapter, entitled *Elements of the Theory of Fields*, deals with vector fields, divergence, curl, potentials, flux, and circulation. The total number of problems is 4465. Answers to all problems are given. For the more difficult problems hints or directions are supplied. There are four pages of numerical tables for use with the problems.

Anyone who is quite familiar with the problems and exercises in American texts of the last thirty years or so on calculus, elementary differential equations and closely related material, will find little that is novel or surprising here. The conceptual emphasis is old-fashioned and the demands on technical proficiency are rather high. The number of problems that seem to offer great intellectual challenge or to demand great ingenuity and virtuosity is quite small. A conscientious teacher, looking for welcome variations on familiar themes, may glean some profit from possessing and carefully examining the book. The publication of the book will certainly augment the usefulness of the English edition of Bermant's *A Course of Mathematical Analysis.*

A. E. TAYLOR, University of California, Los Angeles explanation and understanding is enormous. As I wrote in a recent essay, "Man, Machines and Morality" (*Man on Earth*—vol. I/No. 3), the machine has now become a problem more of morality than of technology.

Wiener's "priests of power" and "gadget worshipers" are indeed the quantifiers of "success" and, by direct extension, they are the corrupters of the young.

Socrates took the hemlock for a good deal less.

S P R CHARTER, Olema, California

A Collection of Problems on a Course of Mathematical Analysis. By G. N. Berman. Translated from Russian by D. E. Brown, and edited by I. N. Sneddon. Pergamon Press, New York, 1965. ix+588 pp. $12.50.

This is an English translation of the tenth edition of a standard problem book first published in 1947. According to the book's own statement it is intended for students studying mathematical analysis within the framework of a Technical College course, and it is especially suited for use with the text, *A Course of Mathematical Analysis*, by A. F. Bermant. (For a review of the two-volume English translation of this work see this MAGAZINE 38 (1965) 175–176.) Professor Bermant edited the earlier editions of the *Collection of Problems*, and participated in the work of the revisions, the tenth edition excepted.

There are sixteen chapters. The first fifteen of these follow exactly the subject matter as arranged in the fifteen chapters of the text by Professor Bermant. The sixteenth chapter, entitled *Elements of the Theory of Fields*, deals with vector fields, divergence, curl, potentials, flux, and circulation. The total number of problems is 4465. Answers to all problems are given. For the more difficult problems hints or directions are supplied. There are four pages of numerical tables for use with the problems.

Anyone who is quite familiar with the problems and exercises in American texts of the last thirty years or so on calculus, elementary differential equations and closely related material, will find little that is novel or surprising here. The conceptual emphasis is old-fashioned and the demands on technical proficiency are rather high. The number of problems that seem to offer great intellectual challenge or to demand great ingenuity and virtuosity is quite small. A conscientious teacher, looking for welcome variations on familiar themes, may glean some profit from possessing and carefully examining the book. The publication of the book will certainly augment the usefulness of the English edition of Bermant's *A Course of Mathematical Analysis*.

A. E. TAYLOR, University of California, Los Angeles

PROBLEMS AND SOLUTIONS

Edited by Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in india ink and exactly the size desired for reproduction.

Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROPOSALS

607. *Proposed by Sidney Kravitz, Picatinny Arsenal, Dover, New Jersey.*

Number the faces of a dodecahedron with the numbers 1 through 12. It is easy to place the numbers such that those on any two adjacent pentagonal faces differ by at least 2. Show that it is impossible to place the numbers such that those on any two adjacent faces differ by at least 3.

608. *Proposed by A. A. Gioia and A. M. Vaidya, Texas Technological College.*

Call a positive integer n semi-perfect if the sum of all the square free divisors of n is $2n$. Prove that 6 is the only semi-perfect number.

609. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Solve the following cryptarithm in the decimal system:

$$4 \cdot NINE = 9 \cdot FOUR$$

610. *Proposed by Leon Bankoff, Los Angeles, California.*

According to the Erdös-Mordell theorem,

$$AI + BI + CI \geqq 6r$$

where I is the incenter and r the inradius of triangle ABC, with equality when the triangle is equilateral. Show that

$$AP + BP + CP \geqq 6r$$

where P is any point within triangle ABC.

611. *Proposed by A. Struyk, Paterson, New Jersey.*

A well-known problem is that in which a rectangular sheet of given dimensions (length L, width W) is to have cut from its corners a square (side x) so that, when the resulting figure is folded to form an open box, the box will have maximum volume.

(a) Let it be required to choose integers L and W for which x is rational. Show that proper manipulation of four suitable numbers in arithmetic sequence yields appropriate values for L, W, and x.

(b) Show that there are sheets with integral dimensions which, when cut as specified, yield maximum volume boxes whose dimensions are triangular numbers.

612. *Proposed by M. B. McNeil, University of Missouri at Rolla.*

The integral

$$I_1 = \frac{1}{\pi^3}\int_0^\pi \int_0^\pi \int_0^\pi \frac{du\, dv\, dw}{1 - \cos u \cos v \cos w}$$

occurs in the study of ferromagnetism and in the study of lattice vibrations. Prove that

$$I_1 = (4\pi^3)^{-1}[\Gamma(1/4)]^4.$$

613. *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada.*

In the isosceles triangle ABC, we have $\angle ABC = \angle ACB = 80°$. With E on side AB, $\angle BCE = 50°$, and with D on side AC, $\angle CBD = 60°$. By purely geometrical considerations, and without use of any trigonometrical techniques, evaluate the angle BDE.

SOLUTIONS

Late Solutions

William E. F. Appuhn, Long Island University: **565**; *Mrs. A. C. Garstang, Boulder, Colorado*: **579**; *Jeffrey L. Johnson, Muncie, Indiana*: **579**; *M. S. Klamkin, Ford Scientific Laboratory*: **584**, **585**; *Douglas Lind, University of Virginia*: **581**.

No Flying Saucer

586. [May, 1965] *Proposed by Maxey Brooke, Sweeny, Texas.*

"Jim Clark told me that he saw a flying saucer," I told Ford.

"You can't believe a word Clark says," Ford answered.

"That's peculiar," I replied with my usual degree of truthfulness, "Clark said just the opposite about you."

What is the probability that Clark saw a flying saucer?

Solution by Leon Bankoff, Los Angeles, California.

If the narrator is telling the truth, the last two statements are inconsistent. If Ford is truthful, Clark lies when commending Ford's veracity. If Ford is lying, we can believe Clark's statement that Ford is truthful. In either case, the contradiction can be resolved by recognizing that the narrator's usual degree of truthfulness is nil. It follows that Clark said nothing about having seen a flying saucer and that the required probability is zero based upon the conversation.

Also solved by the proposer. One incorrect solution was received.

588. [May, 1965] *Proposed by Joseph D. E. Konhauser, HRB-Singer, Inc., State College, Pennsylvania.*

Show that the operators $(D-1)^n \times (D-1)$ and $x(D-1)^{n+1}+n(D-1)^n$ are equivalent for $n=1, 2, 3, \cdots$, where $D \equiv d/dx$.

I. *Solution by K. L. Yocom, South Dakota State University.*

Evidently the first expression should read

$$(D-1)^n x(D-1).$$

Assuming this, the equivalence is easily checked for $n=1$. Proceed by induction and assume the equivalence for n. Then

$$\begin{aligned}(D-1)^{n+1}x(D-1) &= (D-1)x(D-1)^{n+1} + n(D-1)^{n+1}\\ &= D[x(D-1)^{n+1}] - x(D-1)^{n+1} + n(D-1)^{n+1}\\ &= xD(D-1)^{n+1} - x(D-1)^{n+1} + (D-1)^{n+1} + n(D-1)^{n+1}\\ &= x(D-1)^{n+2} + (n+1)(D-1)^{n+1},\end{aligned}$$

and the induction is complete.

II. *Solution by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Let $L(D)$ designate any linear differential operator with variable coefficients (it could even be just a function of x). Then by Leibniz rule for the nth derivative,

$$D^n\{xL(D)\} \equiv xD^nL(D) + nD^{n-1}L(D).$$

Now multiply by $\exp\int pdx$ (p is an arbitrary function of x) and use the exponential shift theorem, i.e.,

$$\exp \int pdxL(D) \equiv L(D-p) \exp \int pdx.$$

This yields:

$$\{(D-p)^n xL(D-p) - x(D-p)^nL(D-p) - n(D-p)^{n-1}L(D-p)\} \exp \int pdx \equiv 0$$

or equivalently

$$(D-p)^n xL(D-p) \equiv x(D-p)^nL(D-p) + n(D-p)^{n-1}L(D-p).$$

The proposed problem corresponds to the special case $L(D)=D$, $p=1$.

Also solved by Mrs. A. C. Garstang, Boulder, Colorado; M. J. Pascual, Watervliet Arsenal, New York; Sidney Spital, California State Polytechnic College; and the proposer.

Inverting Cups

589. [May, 1965] *Proposed by Charles W. Trigg, San Diego, California.*

(a) Show that four upright cups all can be inverted by turning over three at a time in exactly $2n$ moves, $n=2, 3, \cdots$.

(b) The four upright cups can be inverted by turning over two at a time in exactly n moves, $n=2, 3, \cdots$.

(c) One inverted and three upright cups cannot all be inverted by turning over two at a time.

Solution by Douglas Lind, University of Virginia.

(A) It is simple to find a 4-move solution. Now inverting 3 cups $2k$ repeated times will not essentially alter their orientation, so that there always exists a solution with $4+2k$ moves, as desired.

(B) One easily finds a 2- and 3-move solution. As in (A), inverting 2 cups $2k$ repeated times does not change their positions, so that there exist solutions in $2+2k$ and $3+2k$ moves, as desired.

(C) Let an up position be represented by 1, a down position by 0, and $G(P)$ be the sum of the orientation numbers at a position P. Clearly inverting 2 cups does not change the parity of $G(P)$. But (C) implies $G(P)$ changes from 3 to 0, and this contradiction establishes the impossibility of (C).

Also solved by Allan Chuck, San Francisco, California; Michael Goldberg, Washington, D. C.; Erwin Just, Bronx Community College; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania; Stanley Rabinowitz, Far Rockaway, New York; Simeon Reich, Haifa, Israel; Sidney Spital, California State Polytechnic College; K. L. Yocom, South Dakota State University; and the proposer.

Simultaneous Equations

590. [May, 1965] *Proposed by R. J. Cormier, University of Missouri.*

Show that the following set of simultaneous equations has no solution in distinct positive integers:

$$a^3+b^3=c^3+d^3$$

$$a+b=c+d.$$

I. *Solution by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania.*

From $a^3+b^3/a+b=c^3+d^3/c+d$, we have

$$a^2-ab+b^2=c^2-cd+d^2,$$

while squaring $a+b=c+d$ leads to $a^2+2ab+b^2=c^2+2cd+d^2$. Combining, we find $ab=cd$.

Thus:

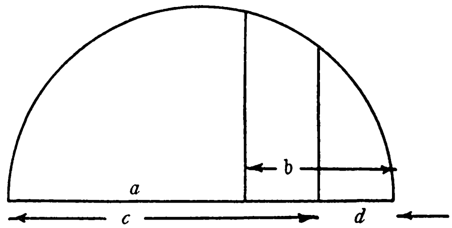
$$(1)\quad a+b=c+d \qquad (2)\quad ab=cd.$$

Let $a\geqq c$ and $b\leqq d$ without loss of generality. Then $a=c+k$ and $b=d-m$ with $k\geqq 0$ and $m\geqq 0$. But (1) implies $m=k$, so that $a=c+k$ and $b=d-k$. Substituting in (2), we obtain $(d-c)k-k^2=0$. Thus, either $k=0$ (which implies $a=c$ and $b=d$) or $k=d-c$, which implies $b=c$ and consequently $a=d$. We have therefore proved the stronger result that the given equations have no solutions in distinct positive real numbers.

II. *Solution by Charles W. Trigg, San Diego, California.*

$$a^3 + 3a^2b + 3ab^2 + b^3 = c^3 + 3c^2d + 3cd^2 + d^3$$
$$a^3 + b^3 = c^3 + d^3$$

Subtracting and simplifying, $3ab(a+b) = 3cd(c+d)$. Now since $a+b=c+d$, then $ab=cd$. These statements are equivalent to stating that the arithmetic means of the two pairs of integers are equal, and that the geometric means of the two pairs are equal. In this event, the pairs must be identical. This is seen easily by referring to the following figure, wherein the radius is the arithmetic mean and the perpendiculars are geometric means.



Also solved by Winifred Asprey, Vassar College; Leon Bankoff, Los Angeles, California; Maxey Brooke, Sweeney, Texas; Allan Chuck, San Francisco, California; Frank Eccles, Phillips Academy, Andover, Massachusetts; Michael Goldberg, Washington, D. C.; Rosemary M. Griffith, Technical Operations Research, Burlington, Massachusetts; Stephen Hoffman, Trinity College, Connecticut; J. A. H. Hunter, Toronto, Canada; Erwin Just, Bronx Community College; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; Douglas Lind, University of Virginia; E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania, Stanley Rabinowitz, Far Rockaway, New York; Simeon Reich, Haifa, Israel; Joel E. Schneider, University of Oregon; David Shannon, University of New Mexico; Sidney Spital, California State Polytechnic College; Michael Tiller, University of Minnesota; John Waddington, Levack, Ontario, Canada; Raymond E. Whitney, Lock Haven State College, Pennsylvania; Dale Woods, Missouri State Teachers College; K. L. Yocom, South Dakota State University; Charles Ziegenfus, Madison College, Virginia; Brother Louis Zirkel, Marist College, New York, and the proposer.

Fibonacci Inequality

591. [May, 1965] *Proposed by M. N. S. Swamy, University of Saskatchewan, Regina, Canada.*

Show that the nth Fibonacci number (defined by $F_1 = F_2 = 1$, $F_n = F_{n+1} + F_{n-2}$) satisfies the inequality

$$\left(\frac{1+\sqrt{5}}{2}\right)^{n-2} < F_n < \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} \quad \text{for } n \geqq 3.$$

I. *Solution by E. L. Magnuson, HRB-Singer, Inc., State College, Pennsylvania.*

By factoring the expression

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

we have

$$F_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}\left(\frac{1-\sqrt{5}}{2}\right) + \cdots + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$
$$< \left(\frac{1+\sqrt{5}}{2}\right)^{n-1}$$

since, in the expansion, the negative terms are clearly greater than the positive terms. By combining the first two terms in the expansion, we have

$$F_n = \left(\frac{1+\sqrt{5}}{2}\right)^{n-2} + \left(\frac{1+\sqrt{5}}{2}\right)^{n-3}\left(\frac{1-\sqrt{5}}{2}\right)^2 + \cdots + \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}$$
$$> \left(\frac{1+\sqrt{5}}{2}\right)^{n-2}$$

since, in this form, the positive terms are clearly greater than the negative terms.

II. *Solution by Charles Ziegenfus, Madison College, Virginia.*

Using induction, we shall prove that

$$\alpha^{n-2} < F_n < \alpha^{n-1}, \qquad n \geqq 3,$$

where $\alpha = (1+\sqrt{5})/2$ and α is a root of $x^2 = x+1$.

The statement is obviously true for $n=3$ and $n=4$. Suppose that the statement is true for $n=3, 4, \cdots, k$. In particular

$$\alpha^{k-2} < F_k < \alpha^{k-1}$$
$$\alpha^{k-3} < F_{k-1} < \alpha^{k-2}.$$

Adding we obtain,

$$\alpha^{k-2} + \alpha^{k-3} < F_k + F_{k-1} < \alpha^{k-1} + \alpha^{k-2}.$$

We note that $\alpha^{k+2} = \alpha^{k+1} + \alpha^k$. Therefore, $\alpha^{k-1} < F_{k+1} < \alpha^k$ and the desired result follows.

Note. This problem can be found in Elementary Introduction to Number Theory, Calvin T. Long, D. C. Heath and Co., page 11, problem 11.

Also solved by Marjorie R. Bicknell, San Jose, California; Maxey Brooke, Sweeny, Texas; J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pennsylvania; Mrs. A. C. Garstang, Boulder, Colorado; Michael Goldberg, Washington, D. C.; Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan; James R. Landers, Deer Park, Texas; Douglas Lind, University of Virginia; M. J. Pascual, Watervliet Arsenal, New York; Stanley Rabinowitz, Far Rockaway, New York; Simeon Reich, Haifa, Israel; Frank Servas, Columbia University; Sidney Spital, California State Polytechnic College; Charles W. Trigg, San Diego, California; Howard L. Walton, Falls Church, Virginia; John Wessner, Melbourne, Florida; Raymond E. Whitney, Lock Haven State College, Pennsylvania; K. L. Yocom, South Dakota State University; and the proposer.

Servas found the problem in Niven and Zuckerman, "An Introduction to the Theory of Numbers," p. 93, problem 4. He notes an interesting application is to be found in Uspensky and Heaslet, "Elementary Number Theory," p. 43, in the proof of Lamé's theorem.

A Perfect Square Sum

592. [May, 1965] *Proposed by J. S. Vigder, Defence Research Board of Canada, Ottawa, Canada.*

Determine the values of n for which $\sum_{k=1}^{n} k^5$ is a perfect square.

I. *Solution by Joseph Arkin, Suffern, New York.*

If $n \geqq 1$, we have

$$S_5 = \sum_{k=1}^{n} k^5 = (n^2(n+1)^2(2n^2+2n-1))/12,$$

so that we must find, say,

$$3y^2 = 2n^2 + 2n - 1, \quad \text{for } S_5 \text{ to be a square.}$$

Then, let $n = 3x+1$, where we get

$$y^2 = 6x^2 + 6x + 1. \tag{1}$$

We now solve for (1) by employing Euler's formula; he treated

$$ax^2 + bx + c = y^2,$$

given the solution

$$x = f, \qquad y = m.$$

Set $x = f + uz$, $y = m + vz$. Then $(v^2 - au^2)z = 2auf - 2vm + bu$. Since $a = b = 6$ is positive and not a square, we can make

$$v^2 - au^2 = 1 \quad \text{(a Pell equation)}$$

and obtain an infinite number of integral solutions.

The first few examples are (with $a = b = 6$ and $c = 1$)

$$x = 0, 4, 44, \cdots,$$

$$y = 1, 11, 109, \cdots,$$

so that in the proposed problem (since $n = 3x+1$) we have

$$n = 1, 13, 133, \cdots$$

II. *Solution by Mrs. A. C. Garstang, Boulder, Colorado.*

The formula for the sum is well known, and is

$$\frac{1}{12} n^2(n+1)^2(2n^2+2n-1)$$

(e.g., C. V. Durell, Advanced Algebra, Vol. 1, G. Bell, London, 1932, p. 42, problem 31). If n is any integer, a factor 4 cancels from the numerator and denominator, from either the n^2 or $(n+1)^2$ terms. So we have to make $2n^2+2n-1 = 3m^2$ where m is integral. This may be written

$$(2n+1)^2 = 6m^2 + 3.$$

Thus $2n+1$ must contain a factor 3, and writing $2n+1=3N$ we obtain the equation $3N^2-2m^2=1$. Solutions of this equation may be obtained by the method of continued fractions. Expressing $(2/3)^{1/2}$ as a continued fraction, we obtain

$$\frac{2^{1/2}}{3}=\frac{1}{1+}\;\frac{1}{4+}\;\frac{1}{2+}\;\frac{1}{4+}\;\frac{1}{2+}\cdots$$

and the first, third, fifth, seventh, . . . convergents supply solutions of the equation. The convergents are 1, 4/5, 9/11, 40/49, 89/109, 396/485, 881/1079, . . . and hence the required values of N are 1, 9, 89, 881, . . . and thus the values of n are 1, 13, 133, 1321, . . .

Also solved by Arlo D. Anderson, U. S. Naval Research Laboratory, Washington, D. C.; R. H. Anglin, Danville, Virginia; Maxey Brooke, Sweeney, Texas; L. Carlitz, Duke University; Michael Goldberg, Washington, D. C.; J. A. H. Hunter, Toronto, Ontario, Canada; Erwin Just, Bronx Community College; P. D. Thomas, U. S. Naval Oceanographic Office, Suitland, Maryland; Charles W. Trigg, San Diego, California; K. L. Yocom, South Dakota State University; and the proposer. One incorrect solution and one partial solution were received.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q374 What is the largest number of Senate committees that can be formed with the property that each pair of committees has at least one member in common?

[*Submitted by David L. Silverman*]

Q375 Three nonzero numbers form a geometric progression whose ratio is an integer. If nine is added to the smallest number, an arithmetic progression is formed. Find all possible integral values for the numbers.

[*Submitted by Stanley Rabinowitz*]

Q376 Nine cubes of each color—red, white and blue—all congruent, are to be assembled into a 3 by 3 by 3 cube in such a way that a cube of every color will appear in every row, column and pile. In how many essentially different ways can this be done?

[*Submitted by Charles W. Trigg*]

Q377 Solve the difference equation

$$P_{n+1}-2P_n+(1+x^2)P_{n-1}=0,$$

where $P_0=a(x)$ and $P_1=b(x)$.

[*Submitted by Murray S. Klamkin*]

(Answers on pages 42–43)

ANSWERS

A374 2^{99}. The 2^{99} possible committees on which a given senator might serve clearly meet the condition. If more than 2^{99} committees are chosen, i.e., more than half of the 2^{100} possible committees including the null committee, committee of one, and the entire Senate, then at least two will be "complementary" hence disjoint.

A375 Calling the numbers a, ar, and ar^2 we have

$$a + 9 + ar^2 = 2ar, \quad \therefore \quad r^2 - 2r + 1 = \frac{-9}{a} = (r-1)^2.$$

Hence a must equal -1 or -9 so that $-9/a$ is a perfect square. When $a=-9$ we get $r=2$ so the numbers are $-1, -4, -16$; $-1, 2, -4$; or $-9, -18, -36$. The trick is that the smallest number is not necessarily the first number (in fact, in the three solutions found above, the smallest number is last). If we assume the smallest number is the middle number, we get the equation

$$a + ar^2 = 2ar + 18$$

which leads to

$$(r-1)^2 = 18/a.$$

This gives $a=2$ and $r=-2$ so that the following set is also a solution: $2, -4, 8$.

A376 At least three, with different colors at the centers of the large cubes. The arrangements in the cubes of the Quickie may be used in successive layers, i.e.,

$$\begin{matrix} R & W & B \\ B & R & W \\ W & B & R \end{matrix} \qquad \begin{matrix} W & B & R \\ R & W & B \\ B & R & W \end{matrix} \qquad \begin{matrix} B & R & W \\ W & B & R \\ R & W & B. \end{matrix}$$

Cyclic permutations of the colors gives the two other cubes,

$$\begin{matrix} W & B & R \\ R & W & B \\ B & R & W \end{matrix} \qquad \begin{matrix} B & R & W \\ W & B & R \\ R & W & B \end{matrix} \qquad \begin{matrix} R & W & B \\ B & R & W \\ W & B & R \end{matrix}$$

and

$$\begin{matrix} B & R & W \\ W & B & R \\ R & W & B \end{matrix} \qquad \begin{matrix} R & W & B \\ B & R & W \\ W & B & R \end{matrix} \qquad \begin{matrix} W & B & R \\ R & W & B \\ B & R & W. \end{matrix}$$

Every other apparently different arrangement goes into one of these three by rotation.

A377 Let $xQ_n = P_{n+1} - P_n$, then

$$Q_{n+1} = Q_n - xP_n.$$

Now let $F_n = P_n + iQ_n$, which gives us

$$F_{n+1} = (1 - ix)F_n,$$
$$F_0 = b(x) + i[b(x) - a(x)]/x$$

and

$$F_n = (1 - ix)^n F_0,$$

where P equals the real part of F_n.

(Quickies on page 76)

ON COMPLEMENTING SETS OF NONNEGATIVE INTEGERS

A. M. VAIDYA, Bombay, India

1. DEFINITION. *Two sets A and B of nonnegative integers are called complementing sets if*

(i) $A \cap B = \{0\}$, *and*

(ii) *every positive integer has a unique representation in the form $a+b$ for some $a \in A$, $b \in B$.*

2. Let m be any positive integer. Then the following pairs of sets are complementing.

Example 1. $A = \{0, 1, 2, 3, 4, \cdots\}$, $B = \{0\}$.

Example 2. $A = \{0, 1, 2, \cdots, m-1\}$, $B = \{0, m, 2m, 3m, \cdots\}$.

Example 3. $A = \{2km, 2km+1, \cdots, 2km+m-1 \mid k=0, 1, 2, \cdots\}$, $B = \{0, m\}$.

Example 4. $A = \{0, 1, 2, \cdots, m-1, 2m, 2m+1, \cdots, 2m+m-1\}$, $B = \{0, m, 4m, 5m, 8m, 9m, \cdots, 4km, (4k+1)m, \cdots\}$.

In this note, we prove some of the properties of complementing sets suggested by these examples.

3. In the following, we assume that A and B are complementing sets. We exclude the trivial Example 1 of Section 2 from consideration. That is, we assume that both A and B contain positive integers. If an integer n is contained in A or B, then we say, "n appears." Obviously, 1 appears. We assume that $1 \in A$. We let m denote the least positive integer in B. Then $0, 1, 2, \cdots, m-1$ all appear and are in A; 0 and m appear and are in B. It is also clear that the integers $m+1, m+2, \cdots, 2m-1$ do not appear.

4. LEMMA. *For an integer q and an integer r satisfying $1 \leqq r \leqq m-1$, if the integer $qm+r$ appears then so does qm and both are in A. Also if $qm \in A$, then $qm+r \in A$ for each r satisfying $1 \leqq r \leqq m-1$.*

Proof. We prove the lemma by induction on q. The results are clearly true for $q=0$. Suppose they are true up to $q-1$. Notice that this assumption implies that all integers in B which are less than qm are multiples of m.